EXIT FUNCTIONS OF SOFT ITERATIVE MIMO DETECTORS AND THEIR APPLICATION TO CAPACITY-APPROACHING CODING

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ABSTRACT

Design of MIMO capacity-approaching codes relies on matching the EXtrinsic Information Transfer (EXIT) functions of the code and the detector. We focus on List Sphere Detection (LSD) because it presents the best performance/complexity tradeoff. We obtain the EXIT functions of the most representative LSDs: Maximum Likelihood (ML) and Maximum *A Posteriori* (MAP). We then utilize these EXIT functions to design good IRA and SCLDGM codes that perform close to the capacity limits.

Index Terms— MIMO systems, Demodulation, Codes

1. INTRODUCTION

Optimum detection in MIMO systems increases exponentially with the number of transmitting antennas and the constellation size. Several suboptimum detectors have been proposed to enable practical detection, being the most representative MMSE filtering with Soft Interference Cancellation (MMSE-SIC) [1] and those based on List Sphere Detection (LSD) [2, 3].

Low-Density Parity-Check (LDPC) codes are recognized as the best coding scheme to approach the capacity of MIMO channels. Due to their excellent compromise between performance and complexity, we focus on two subclasses of LDPC codes, namely, Irregular Repeat-Accumulate (IRA) [4] and Serially-Concatenated Low-Density Generator Matrix (SCLDGM) [5]. Most recent insights in the design of MIMO capacity-approaching codes suggest that matching the code and the detector EXtrinsic Information Transfer (EXIT) functions attains best performance. In the literature, there exists very little references on MIMO code design considering suboptimum detection. One of the few examples is the design of IRA codes employing MMSE-SIC detection (but not LSD) explained in [6].

In this work we focus on the obtention of EXIT functions for LSD in MIMO systems because this detector presents the best performance/complexity tradeoff in most cases. We consider the most prominent versions of LSD: Maximum Likelihood (ML) [2] and Maximum *A Posteriori* (MAP) [3]. EXIT analysis of MAP LSD is rather involved because positive feedback of the *a priori* information makes the resulting EXIT function useless for code design. We overcome this limitation by resorting to the LSD that maximizes the extrinsic probability, instead of the likelihood or the *a posteriori* probability as done in ML and MAP LSDs, respectively. We have designed rate-$1/2$ IRA and SCLDGM codes for both ML and MAP LSDs and for the MMSE-SIC detector, showing their ability to approach the capacity limits. Interestingly, the code degree profiles obtained for each type of detector are different.

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2. SOFT ITERATIVE MIMO DETECTORS

Let us consider an $n_t \times n_r$ MIMO transmission system where an information sequence $u[k] = [u_1, u_2, \ldots, u_K]$ is encoded with a rate $R = K/N$ binary code to produce the coded sequence $c[k] = [c_1, c_2, \ldots, c_N]$. Each group of M_c coded bits is Gray-mapped into a constellation symbol. These symbols are then assigned to transmitting antennas on a serial-to-parallel basis, resulting in the sequence of transmitted vectors $\mathbf{x}[k]$, $k = 1, 2, ..., L$ with $L = N/(n_t M_c)$. After transmission through the MIMO channel, the signals at reception can be written as

$$\mathbf{y}[k] = \mathbf{H}[k]\mathbf{x}[k] + \mathbf{n}[k] \quad k = 1, 2, \ldots, L \tag{1}$$

where $\mathbf{H}[k]$ is the $n_r \times n_t$ ergodic Rayleigh MIMO channel matrix whose entries, $h_{ij}[k] \sim \mathcal{CN}(0, 1)$, are spatial and temporally uncorrelated. The components $n_i \sim \mathcal{CN}(0, N_0)$ of the noise vector, $\mathbf{n}[k]$, are also spatial and temporally uncorrelated. Denoting by E_s the total energy in each transmitted vector, $\mathbf{x}[k]$, and taking into account that it carries $Rn_t M_c$ information bits, the E_b/N_0 at reception is

$$\frac{E_b}{N_0} = \frac{n_r}{R n_t M_c} \frac{E_s}{N_0} \tag{2}$$

being E_s/N_0 the signal-to-noise ratio (SNR) per receiving antenna.

Decoding of capacity-approaching codes (e.g. LDPC) is performed by applying the Sum-Product Algorithm (SPA). This algorithm takes as input the bit channel Log-Likelihood Ratios (LLRs), L_k^c, given by (to simplify notation, we drop index $[k]$)

$$L_k^c = \log \frac{P(\mathbf{y}|c_k = 1)}{P(\mathbf{y}|c_k = 0)} = \log \frac{P(c_k = 1|\mathbf{y})}{P(c_k = 0|\mathbf{y})} - \underbrace{\log \frac{P(c_k = 1)}{P(c_k = 0)}}_{L_k} \tag{3}$$

where subindex $k = 1, 2, ..., n_t M_c$ refers to the bits carried by the symbol vector $\mathbf{x}$ and L_k is the *a priori* LLR of bit c_k. In *A Posteriori* Probability (APP) detection, LLRs are calculated according to their exact expression given by

$$L_k^c = \log \frac{\displaystyle\sum_{\mathbf{x} \in X_k^+} \exp\left(-\frac{\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2}{N_0} + \sum_{i=1}^{n_t M_c} x_i \frac{L_i}{2}\right)}{\displaystyle\sum_{\mathbf{x} \in X_k^-} \exp\left(-\frac{\|\mathbf{y} - \mathbf{H}\mathbf{x}\|^2}{N_0} + \sum_{i=1}^{n_t M_c} x_i \frac{L_i}{2}\right)} - L_k \tag{4}$$

where $x_i = 2c_i - 1$ and X_k^+ and X_k^- represent the set of all transmitted symbol vectors $\mathbf{x}$ when $c_k = 1$ and $c_k = 0$, respectively. Notice that the complexity of APP detection grows exponentially in

$n_t M_c$, becoming infeasible in many cases. To overcome this limitation, several suboptimum approaches have been proposed, being the most prominent ones, in terms of performance and complexity, List Sphere Detection (LSD) and MMSE filtering with Soft Interference Cancellation (MMSE-SIC).

The basic premise of LSD is to approximate the summations over $\mathbf{x} \in X_k^+$ (resp. $\mathbf{x} \in X_k^-$) in Eq. (4) using a list of the N_{cand} most relevant vectors $\mathbf{x}$, referred to as candidates. Notice that the terms in these summations are the logarithm of the APP of a transmitted vector $\mathbf{x}$

$$\log P(\mathbf{x}|\mathbf{y}) \propto -\frac{1}{N_0}\|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2 + \sum_{i=1}^{n_t M_c} x_i \frac{L_i}{2} \qquad (5)$$

where $\propto$ represents equality up to an additive constant. The candidate list is constructed using a modified version of the Sphere Detector (SD), which efficiently finds the constellation points inside a hypersphere centered on an initial estimate of the transmitted vector. We focus on two different criteria for constructing the candidate list, namely, Maximum Likelihood (ML) [2] and Maximum *A Posteriori* (MAP) [3]. In ML LSD, the candidate list is built using the most likely vectors, i.e. those that maximize the term in Eq. (5) corresponding to the likelihood, i.e.,

$$\log P(\mathbf{y}|\mathbf{x}) \propto -\frac{1}{N_0}\|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2 \qquad (6)$$

In contrast, the MAP LSD builds the list of the N_{cand} vectors with the highest APP given by Eq. (5).

At a first glance, MAP LSD seems preferable because it actually finds the most significant terms in the computation of the channel LLRs. However, when used in an iterative receiver, it has to perform the complete tree search each time new *a priori* bit LLRs are available from other receiver stages. On the other hand, the complexity of ML LSD is smaller, since it carries out only once the search for the most likely vectors and stores the corresponding likelihoods for their use in successive iterations. If we keep the number of operations fixed when comparing both methods, ML LSD allows the utilization of a larger candidate list, whereas the size of the candidate list in MAP LSD can be reduced as the iterative detection process evolves. However, it is important to note that Eq. (4) computes extrinsic probabilities, while both ML and MAP LSDs select candidates that do not maximize the extrinsic probability but an alternative (although related) criterion. In the following section we will see that this has important consequences not only in performance but in the detector characterization for code design.

MMSE-SIC represents an alternative detection approach where complexity of channel LLR computation is kept at a reasonable level by spatially decoupling the detection process. We refer the reader to [6] for a detailed explanation of this detector. When compared to LSD, MMSE-SIC detection has the advantage of being more scalable and, thus, more appropriate for large numbers of antennas and/or constellations sizes.

3. EXIT FUNCTIONS OF MIMO DETECTORS

EXtrinsic Information Transfer (EXIT) functions play a key role in the design of soft iterative receivers. An EXIT function relates the mutual information of extrinsic output bit LLRs, I_e, versus that of *a priori* input bit LLRs, I_a, in each component module. We next discuss the obtention of EXIT functions for the MIMO detectors previously described. Since extrinsic output LLRs (see Eq. (4)) are the result of an involved operation, it is not possible to obtain their actual mutual information analytically and we resort to Monte Carlo methods. For a fixed E_s/N_0, we first simulate the transmission through a MIMO channel. Next, the mutual information of *a priori* messages, I_a, is set to a certain value. *A priori* bit messages with such a mutual information are easily generated by considering them as output LLRs of a Binary-Input AWGN (BIAWGN) channel. Finally, both the received samples and the generated *a priori* bit LLRs are fed into the MIMO detector to produce a set of output LLRs whose mutual information, I_e, can be numerically evaluated.

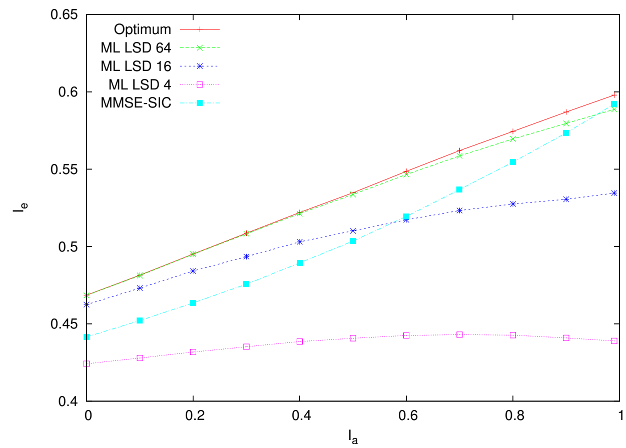

Fig. 1. EXIT function of optimum, ML LSD and MMSE-SIC MIMO detectors for 4×4 QPSK and $E_s/N_0 = 2.0$ dB

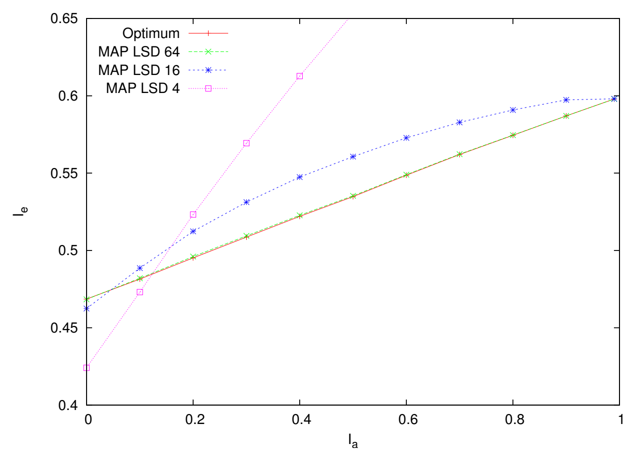

Fig. 2. Output mutual information of MIMO MAP LSD for 4×4 QPSK and $E_s/N_0 = 2.0$ dB

Let us consider a 4×4 QPSK MIMO system operating at $E_s/N_0 = 2$ dB. Figure 1 compares the EXIT functions of the ML LSD with that of the optimum for different numbers of candidates. Notice the ability of the ML LSD to approach the optimum detector when the number of candidates is sufficiently large. In this MIMO configuration, this is accomplished when $N_{cand} = 64$, which represents 25% of the total number of possible candidates, 256. However, the behaviour of ML LSD severely degrades when an insufficient number of candidates is selected. Figure 1 also plots the EXIT function of the MMSE-SIC detector. Note the slope and concavity of this EXIT function, which hinders code design for this specific detector.

The mutual information of output messages for the MAP LSD is plotted in Figure 2. It is apparent that it cannot be considered as a true EXIT function since it exceeds that of the optimum detector. This is because MAP LSD presents positive feedback. Although the *a priori* LLR of the bit being processed, L_k, is substracted in Eq. (4) to obtain the extrinsic channel LLR, it is implicitly considered in the construction of the candidate list. Note that a high *a priori* LLR of the bit being considered leads to the inclusion in the list of those candidates for which that bit has the value suggested by the *a priori* bit LLR, even though their extrinsic probability (which does not count the *a priori* probability) might be very low. ML LSD does not suffer from this positive feedback since it does not take into account any *a priori* probability, neither that of the bit being considered nor the rest. The effect of positive feedback is more pronounced as the size of the candidate list gets smaller and, clearly, is not compensated by just subtracting the *a priori* LLR. Although the performance penalty produced by positive feedback may not be severe, it has important consequences regarding code optimization. Since it is only the extrinsic information transfer what determines decoding convergence, it is necessary to cancel the contribution of the *a priori* information from the output LLRs when computing the actual EXIT function of this detector. However, this cannot be done because *a priori* information is implicitely introduced by the particular way the candidate list is built. To overcome this limitation, we propose to measure the EXIT function of what we term as the Extrinsic LSD.

Rewriting Eq. (4) as

$$L_{\mathrm{ch}} = \log \frac{\displaystyle\sum_{\mathbf{x}\in X_k^+} \exp\left(-\frac{1}{N_0}\|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2 + \sum_{i=1,i\neq k}^{n_t M_c} x_i \frac{L_i}{2}\right)}{\displaystyle\sum_{\mathbf{x}\in X_k^-} \exp\left(-\frac{1}{N_0}\|\mathbf{y}-\mathbf{H}\mathbf{x}\|^2 + \sum_{i=1,i\neq k}^{n_t M_c} x_i \frac{L_i}{2}\right)} \quad (7)$$

clearly suggests to build a separate list for each bit in a symbol vector. This means $n_t M_c$ candidate lists for computing the bit channel LLRs in a symbol vector, instead of only one as in MAP LSD. The complexity of this approach is $n_t M_c$ times higher than that of MAP LSD, but its expected performance is better. In addition, since it does not suffer from positive feedback, its *true* EXIT function can be computed, enabling code design. However, note that if MAP LSD is used with a high number of candidates, the effect of the *a priori* LLR is less severe. This motivates us to use MAP LSD in the final implementation shown in Section 4, even though code design is performed using the EXIT function of the Extrinsic LSD. Figure 3 plots the EXIT function of the Extrinsic LSD for different sizes of the candidate lists. It is apparent from this figure the superior performance of the Extrinsic LSD when compared with the ML LSD (cf. Fig 1), especially for a very low number of candidates.

4. RESULTS

Once we have obtained the EXIT functions of the MIMO detectors, we proceed to code design. Due to their excellent compromise between performance and complexity, we focus on Irregular Repeat-Accumulate (IRA) [4] and Serially-Concatenated Low-Density Generator Matrix (SCLDGM) codes [5]. We follow the procedure described in [7] which is based on Density Evolution under the symmetric Gaussian assumption of message densities, but where tracking is carried out using EXIT functions instead of a density param-

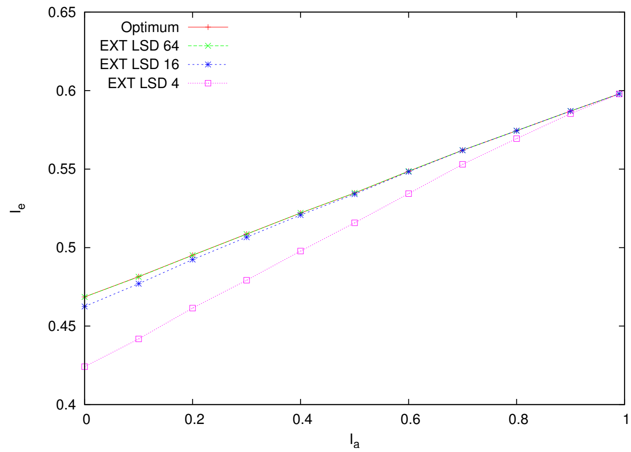

Fig. 3. EXIT function of MIMO Extrinsic LSD for 4×4 QPSK and $E_s/N_0 = 2.0$ dB

eter. Table 1 summarizes the best rate-1/2 systematic codes found for different MIMO configurations and detectors.

We have carried out computer simulations to illustrate the performance of the designed codes. Fig. 4 plots the Bit-Error Rate (BER) vs E_b/N_0 for a 4×4 QPSK MIMO system. The constrained-input capacity limit (CCL) is located at $E_b/N_0 = 1.5$ dB. For a target BER of 10^{-4}, IRA and SCLDGM codes designed for optimum detection exhibit similar performance, within 0.5 dB of the capacity limit. Small degradation in performance (≤ 0.1 dB) is observed when MAP LSD with 64 candidates is employed. However, using MMSE-SIC detection incurs in a 0.35 dB performance loss for SCLDGM codes and 0.45 dB for IRA codes.

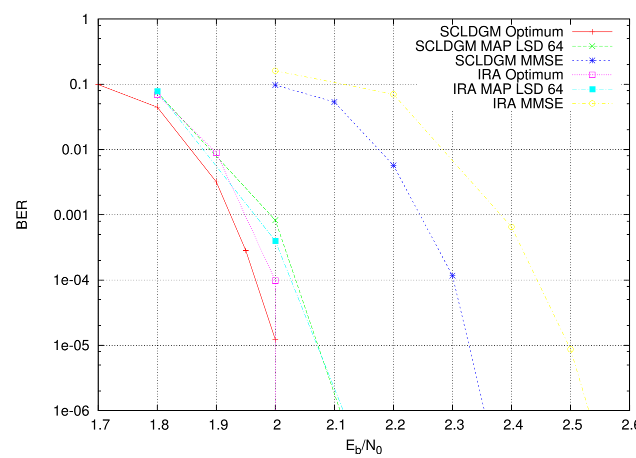

Fig. 4. BER vs E_b/N_0 for 4×4 QPSK. CCL at $E_b/N_0 = 1.5$ dB. Block length $N = 100000$ coded bits.

Fig. 5 shows the results for 16-QAM, where optimum detection is infeasible. SCLDGM codes designed for MAP LSD with 64 candidates ($\approx 0.01\%$ of the total number of candidates, 2^{16}) achieve the best performance, within 1.35 dB of the CCL. Although the predicted thresholds for IRA and SCLDGM codes are very similar, the actual performance of IRA codes is worse (at $E_b/N_0 = 6.05$ dB). This is because the actual (empirically measured) and the theoretical (according to [8, 7] EXIT functions are different when there is a finite number of iterations at the decoder. In order to keep complexity

SCLDGM CODES								
Modul.	Detector	$p(\%)$	$d_u^{f^1}$	$d_u^{f^2}$	$d_{p^1}^{f^2}$	Thresh (dB)	UCL	CCL
QPSK	Optimum, EXT 64	2%	3	5	15	1.8	1.2	1.5
	MMSE-SIC	1.5%	3	5	42	2.3		
16QAM	ML 512/256/128/64	2.5%	3	5	20	5.1/5.1/5.3/5.5	3.8	4.1
	EXT 256/128/64/32	3.5%	3	4	16	4.8/4.9/4.9/5.0		
	MMSE-SIC	2.5%	3	4	36	6.1		
64QAM	ML 256/128/64	2.5%	3	5	20	8.9/9.2/9.7	6.3	-
	EXT 128/64/32	2.5%	3	4	24	7.8/7.9/8.1		
	MMSE-SIC	1.5%	5	5	40	10.3		

IRA CODES						
Modul.	Detector	d_v	a_v	Thresh(dB)	UCL	CCL
QPSK	Opt, EXT 64	3, 10, 47	0.78, 0.16, 0.06	1.8	1.2	1.5
	MMSE-SIC	3, 11, 42	0.82, 0.10, 0.08	2.3		
16QAM	ML 256	3, 7, 39	0.82, 0.10, 0.08	5.0	3.8	4.1
	EXT 64	3, 14, 60	0.88, 0.08, 0.04	4.8		
	MMSE-SIC	3, 12, 46	0.88, 0.06, 0.06	6.1		
64QAM	ML 256	3, 8, 37	0.76, 0.18, 0.06	8.8	6.3	-
	EXT 64	3, 14, 59	0.92, 0.04, 0.04	7.9		
	MMSE-SIC	3, 16, 50	0.86, 0.08, 0.06	10.2		

Table 1. Degree profiles of rate $1/2$ optimized SCLDGM and IRA codes for a 4×4 MIMO channel. "Thresh" stands for the EXIT analysis convergence threshold. "UCL" and "CCL" are, respectively, the Unconstrained-input and Constrained-input Ergodic Capacity Limits.

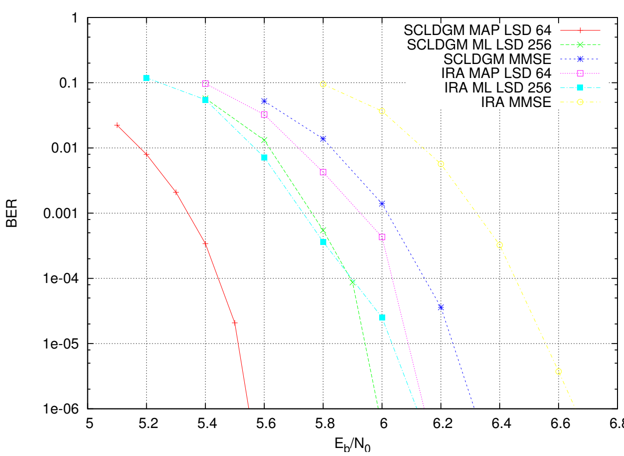

Fig. 5. BER vs E_b/N_0 for 4×4 16-QAM. CCL at $E_b/N_0 = 4.1$ dB. Block length $N = 20000$ coded bits.

at a low level, codes were designed assuming that 10 iterations are performed at the decoder for each iteration at the detector. In this case we have observed that, for IRA codes, the actual and theoretical EXIT functions are considerably different. Notice the robustness of SCLDGM codes since they do not experience such a performance degradation. Performance of ML LSD with 256 candidates is similar for both SCLDGM and IRA codes, at $E_b/N_0 = 5.9$ dB. Finally, the MMSE-SIC detector exhibits the worst performance although, again, SCLDGM codes are better suited to this detector.

5. CONCLUSIONS

We have obtained the EXIT functions of several suboptimum MIMO detectors, namely, MMSE-SIC, ML LSD and MAP LSD. This latter detector performs the best but, due to its positive feedback, its EXIT function cannot be directly computed. We overcome this difficulty by introducing the Extrinsic LSD. The resulting EXIT functions have been used to design good IRA and SCLDGM codes that perform close to the capacity limits.

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