DELAY-LIMITED THROUGHPUT OF SPACE-TIME CODES WITH CHANNEL ESTIMATION ERRORS

Ravi Narasimhan

Department of Electrical Engineering
University of California, Santa Cruz, CA 95064, USA
Email: ravi@soe.ucsc.edu

ABSTRACT

Lower bounds are computed on the throughput of packet-switched space-time coded systems with delay constraints. The delay constraints are accounted for by defining a delay-limited throughput as the instantaneous throughput of each packet that can be maintained with a certain probability. The analysis assumes quasi-static and independent fading across packets, packet retransmissions until success, and negligible overhead for acknowledgment packets and guard times. Upper bounds on the outage probabilities of orthogonal space-time block codes (OSTBC) and spatial multiplexing with horizontal encoding (SM-HE) are computed in the presence of channel estimation errors to obtain lower bounds on the delay-limited throughput. The target physical-layer data rate is selected at each signal-to-noise ratio (SNR) to maximize the lower bounds on the throughput. The results indicate that higher throughput is achieved using OSTBC at low SNR and SM-HE at high SNR. Furthermore, the loss of throughput due to channel estimation errors is quantified.

***Index Terms*—** MIMO systems, packet radio, optimization methods, fading channels, information theory

1. INTRODUCTION

Many wireless systems, such as wireless local area networks (WLANs), operate in quasi-static channels, where the fading is constant for one coding block (or packet) and varies independently between packet transmissions. In such slow fading scenarios, traffic delay constraints often do not permit averaging over several channel states to select user data rates. Thus, the probability of exceeding the delay constraint is a key performance metric for real-time traffic.

The probability of exceeding a delay constraint is equal to the probability that the instantaneous throughput of each packet is less than a target value; this value is defined as the delay-limited throughput in this paper. In contrast to ergodic throughput, the delay-limited throughput represents the user throughput that can be maintained with a certain probability in quasi-static channels. For packet-switched systems such as WLANs, the delay-limited throughput can be determined from the physical-layer packet error rate (PER) and the retransmission policy. In this paper, a packet is assumed to be retransmitted until it is successful. Each retransmission experiences independent quasi-static fading. The PER is defined assuming no channel state information (CSI) at the transmitter and capacity-achieving codes applied per packet. Under these conditions, the packet error rate is equal to the channel outage probability [1], defined as the probability that the target physical-layer data rate exceeds the mutual information, conditioned on the fading realization, between the transmitted and received signals.

In this paper, lower bounds on the delay-limited throughput are obtained using upper bounds on the channel outage probability with channel estimation errors for two classes of space-time codes: orthogonal space-time block codes (OSTBC) [2, 3] and spatial multiplexing with horizontal encoding (SM-HE) [1]. The target physical-layer data rate is selected at each signal-to-noise ratio (SNR) to maximize the throughput lower bounds. The data rate that maximizes throughput at each SNR corresponds to the optimal rate adaptation strategy for packet-switched, space-time coded systems in quasi-static fading.

The remainder of the paper is organized as follows. The system model is given in Section 2. In Section 3, a lower bound on the delay-limited throughput as a function of the channel outage probability is derived. Section 4 presents upper bounds on the channel outage probability for OSTBC and SM-HE with channel estimation errors. Numerical results for maximum throughput lower bounds are given in Section 5. Conclusions are given in Section 6.

2. SYSTEM MODEL

Consider a narrowband space-time coded system with M_T transmit and M_R receive antennas. A quasi-static i.i.d. Rayleigh fading channel is assumed with independent channel realizations across packet transmissions. The $M_R \times 1$ received vector $\mathbf{y}$ for a single transmission can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n} \tag{1}$$

where $\mathbf{x}$ is the $M_T \times 1$ transmitted vector, $\mathbf{n}$ is a $M_R \times 1$ complex additive white Gaussian noise (AWGN) vector with covariance $N_0\mathbf{I}_{M_R}$, and $\mathbf{H}$ is the $M_R \times M_T$ channel matrix of i.i.d. complex Gaussian random variables with zero mean and unit variance. Here, $\mathbf{I}_m$ denotes the $m \times m$ identity matrix. With capacity-achieving codes applied per packet, the PER is equal to the channel outage probability, given by $P_{\text{out}} = \Pr[I < R]$, where I denotes the mutual information between transmitted and received signals (conditioned on the channel realization) and R is the target physical-layer data rate.

With channel estimation errors, the channel matrix can be written as

$$\mathbf{H} = \hat{\mathbf{H}} + \tilde{\mathbf{H}} \tag{2}$$

where $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$ are i.i.d. Gaussian matrices of the channel estimate and estimation error, respectively. In this paper, linear minimum mean-square error (LMMSE) estimation is assumed with orthogonal training symbols to obtain the channel estimates [4]. Under this assumption, the entries of $\tilde{\mathbf{H}}$ and $\hat{\mathbf{H}}$ have variance $\mathcal{E}_{\tilde{h}}$ and $(1 - \mathcal{E}_{\tilde{h}})$, respectively, where $\mathcal{E}_{\tilde{h}}$ is assumed to be known at the receiver. Now, (1) in the presence of channel estimation errors becomes

$$\mathbf{y} = \hat{\mathbf{H}}\mathbf{x} + \tilde{\mathbf{H}}\mathbf{x} + \mathbf{n}. \tag{3}$$

For OSTBC, consider a codeword with T time slots, M data symbols and spatial code rate $r_s = M/T$. After appropriate receiver processing (e.g., stacking and conjugation for the Alamouti code), the $M_R T \times 1$ effective received vector $\mathbf{y}'_{\text{OSTBC}}$ can be written as

$$\mathbf{y}'_{\text{OSTBC}} = \hat{\mathbf{H}}_{\text{eff}}\mathbf{s} + \tilde{\mathbf{H}}_{\text{eff}}\mathbf{s} + \mathbf{n}' \tag{4}$$

where $\hat{\mathbf{H}}_{\text{eff}}$ and $\tilde{\mathbf{H}}_{\text{eff}}$ are $M_R T \times M$ effective channel matrices corresponding to $\hat{\mathbf{H}}$ and $\tilde{\mathbf{H}}$, respectively, $\mathbf{s}$ is the $M \times 1$ vector of data symbols, and $\mathbf{n}'$ is a $M_R T \times 1$ AWGN vector with covariance $N_0\mathbf{I}_{M_R T}$. Independent data symbols are transmitted such that $E[\mathbf{s}\mathbf{s}^*] = (E_s/M_T)\mathbf{I}_M$, where $E[\cdot]$ denotes expectation, E_s is the total transmit energy per time slot and * denotes conjugate transposition. Note that $\rho = E_s/N_0$ is the average SNR per receive antenna. Each component of $\mathbf{s}$ is independently detected after pre-multiplying $\mathbf{y}'_{\text{OSTBC}}$ by $\hat{\mathbf{H}}^*_{\text{eff}}$ [5]. Note that $\hat{\mathbf{H}}^*_{\text{eff}}\hat{\mathbf{H}}_{\text{eff}} = \|\hat{\mathbf{H}}\|_F^2\mathbf{I}_M$ where $\|\cdot\|_F$ denotes the Frobenius norm. Hence,

$$\begin{aligned}\mathbf{z}_{\text{OSTBC}} &= \hat{\mathbf{H}}^*_{\text{eff}}\mathbf{y}'_{\text{OSTBC}} \\ & \|\hat{\mathbf{H}}\|_F^2\mathbf{s} + \hat{\mathbf{H}}^*_{\text{eff}}\tilde{\mathbf{H}}_{\text{eff}}\mathbf{s} + \hat{\mathbf{H}}^*_{\text{eff}}\mathbf{n}'.\end{aligned} \tag{5}$$

Component-wise detection is then performed based on $\mathbf{z}_{\text{OSTBC}}$.

For SM-HE, $M_R \geq M_T$, and the transmitted vector $\mathbf{x}$ contains independent data symbols with covariance $(E_s/M_T)\mathbf{I}_{M_T}$. Consider a LMMSE receiver in which a receive matrix $\mathbf{G}$ pre-multiplies the received vector $\mathbf{y}$ in (3) to obtain a $M_T \times 1$ vector $\mathbf{z}_{\text{SM-HE}}$. One can show that the LMMSE matrix $\mathbf{G}$ is given by

$$\mathbf{G} = \left[\hat{\mathbf{H}}^*\hat{\mathbf{H}} + M_T\left(\mathcal{E}_{\tilde{h}} + \frac{1}{\rho}\right)\mathbf{I}_{M_T}\right]^{-1}\hat{\mathbf{H}}^*. \tag{6}$$

Hence,

$$\begin{aligned}\mathbf{z}_{\text{SM-HE}} &= \mathbf{G}\mathbf{y} \\ &= \mathbf{G}\hat{\mathbf{H}}\mathbf{x} + \mathbf{G}\tilde{\mathbf{H}}\mathbf{x} + \mathbf{G}\mathbf{n}.\end{aligned} \tag{7}$$

Component-wise detection is then performed based on $\mathbf{z}_{\text{SM-HE}}$.

3. LOWER BOUND ON DELAY-LIMITED THROUGHPUT

In this section, a lower bound on the delay-limited throughput is derived assuming each packet is retransmitted until success. Since the PER for each packet is P_{out} and the channel varies independently between packet transmissions, the probability that N transmissions are needed for success is given by $P_{\text{out}}^{N-1}(1 - P_{\text{out}})$. For a physical-layer data rate of R and N transmissions until the packet is successful, the throughput of each packet is R/N if overhead due to acknowledgments and guard times is negligible. Hence, the delay-limited throughput G_t that can be maintained with probability $(1 - P_t)$ is defined by $\Pr[R/N \leq G_t] = P_t$. Therefore,

$$\begin{aligned}P_t &= \Pr[N \geq R/G_t] \\ &= \sum_{m=\lceil R/G_t \rceil}^{\infty}(1 - P_{\text{out}})P_{\text{out}}^{m-1} \\ &= P_{\text{out}}^{\lceil R/G_t \rceil - 1}.\end{aligned} \tag{8}$$

Thus, $\lceil R/G_t \rceil = 1 + \ln P_t / \ln P_{\text{out}}$, and a lower bound on G_t is given by

$$G_t \geq \frac{R}{1 + \frac{\ln P_t}{\ln P_{\text{out}}}}. \tag{9}$$

Note that the above analysis can be easily extended to account for overhead due to acknowledgments and guard times.

4. UPPER BOUNDS ON OUTAGE PROBABILITY

In this section, upper bounds on P_{out} are determined for OSTBC and SM-HE with channel estimation errors. These bounds are obtained by regarding the terms in (5) and (7) with the channel estimation error matrices as additional i.i.d. Gaussian noise [6, 4]. Since component-wise detection is performed, the relevant mutual information given the channel estimate is computed using the signal-to-interference-plus-noise ratio (SINR) for each data symbol in the space-time code. In the following subsections, OSTBC and SM-HE are discussed separately.

4.1. OSTBC

From (5), the SINR of the i-th component for OSTBC is given by

$$\mathrm{SINR}_{i,\mathrm{OSTBC}} = \frac{E_s/M_T}{[\mathbf{R}_{ee,\mathrm{OSTBC}}]_{i,i}},\ i = 1, \ldots, M \qquad (10)$$

where

$$\mathbf{R}_{ee,\mathrm{OSTBC}} = E\left[\left(\frac{\mathbf{z}_{\mathrm{OSTBC}}}{\|\hat{\mathbf{H}}\|_F^2} - \mathbf{s}\right)\left(\frac{\mathbf{z}_{\mathrm{OSTBC}}}{\|\hat{\mathbf{H}}\|_F^2} - \mathbf{s}\right)^*\right]. \qquad (11)$$

Note that

$$\frac{\mathbf{z}_{\mathrm{OSTBC}}}{\|\hat{\mathbf{H}}\|_F^2} - \mathbf{s} = \frac{\hat{\mathbf{H}}_{\mathrm{eff}}^* \tilde{\mathbf{H}}_{\mathrm{eff}}}{\|\hat{\mathbf{H}}\|_F^2}\mathbf{s} + \frac{\hat{\mathbf{H}}_{\mathrm{eff}}^*}{\|\hat{\mathbf{H}}\|_F^2}\mathbf{n}' \qquad (12)$$

and $E[\tilde{\mathbf{H}}_{\mathrm{eff}}\tilde{\mathbf{H}}_{\mathrm{eff}}^*] = M_T \mathcal{E}_{\tilde{h}} \mathbf{I}_{M_R T}$. Hence,

$$\mathbf{R}_{ee,\mathrm{OSTBC}} = \frac{E_s \mathcal{E}_{\tilde{h}} + N_0}{\|\hat{\mathbf{H}}\|_F^2}\mathbf{I}_M \qquad (13)$$

and

$$\mathrm{SINR}_{i,\mathrm{OSTBC}} = \frac{\rho}{M_T(1+\rho\mathcal{E}_{\tilde{h}})}\|\hat{\mathbf{H}}\|_F^2. \qquad (14)$$

Therefore, a lower bound on the mutual information for OSTBC conditioned on the channel estimate is given by

$$\begin{aligned} I_{\mathrm{OSTBC}} &\geq r_s \log_2(1 + \mathrm{SINR}_{i,\mathrm{OSTBC}}) \\ &= r_s \log_2\left(1 + \frac{\rho}{M_T(1+\rho\mathcal{E}_{\tilde{h}})}\|\hat{\mathbf{H}}\|_F^2\right). \end{aligned} \qquad (15)$$

Note that $\|\hat{\mathbf{H}}\|_F^2 = (1-\mathcal{E}_{\tilde{h}})X$ where X is a gamma random variable with parameters $(M_R M_T, 1)$. With this definition and from (15), an upper bound on the outage probability is given by

$$\begin{aligned} P_{\mathrm{out,OSTBC}} &= \Pr[I_{\mathrm{OSTBC}} < R] \\ &\leq \Pr\left[X < \frac{M_T(1+\rho\mathcal{E}_{\tilde{h}})}{\rho(1-\mathcal{E}_{\tilde{h}})}\left(2^{R/r_s} - 1\right)\right] \\ &= \frac{\gamma\left(M_R M_T, \frac{M_T(1+\rho\mathcal{E}_{\tilde{h}})}{\rho(1-\mathcal{E}_{\tilde{h}})}(2^{R/r_s}-1)\right)}{\Gamma(M_R M_T)} \end{aligned} \qquad (16)$$

where $\gamma(a,x) = \int_0^x t^{a-1}e^{-t}\,dt$ and $\Gamma(a)$ are the incomplete gamma and gamma functions, respectively.

4.2. SM-HE

For SM-HE, the SINR of the i-th component is computed from (7) and is given by

$$\mathrm{SINR}_{i,\mathrm{SM-HE}} = \frac{E_s/M_T}{[\mathbf{R}_{ee,\mathrm{SM-HE}}]_{i,i}} - 1,\ i = 1, \ldots, M_T \qquad (17)$$

where

$$\mathbf{R}_{ee,\mathrm{SM-HE}} = E[(\mathbf{z}_{\mathrm{SM-HE}} - \mathbf{x})(\mathbf{z}_{\mathrm{SM-HE}} - \mathbf{x})^*] \qquad (18)$$

and the "-1" term accounts for the bias of the LMMSE receiver. Note that

$$\begin{aligned} \mathbf{z}_{\mathrm{SM-HE}} - \mathbf{x} &= (\mathbf{G}\hat{\mathbf{H}} - \mathbf{I}_{M_T})\mathbf{x} + \mathbf{G}\tilde{\mathbf{H}}\mathbf{x} + \mathbf{G}\mathbf{n} \\ &= \left[\hat{\mathbf{H}}^*\hat{\mathbf{H}} + M_T\left(\mathcal{E}_{\tilde{h}} + \frac{1}{\rho}\right)\mathbf{I}_{M_T}\right]^{-1} \\ &\quad \cdot \left[-M_T\left(\mathcal{E}_{\tilde{h}} + \frac{1}{\rho}\right)\mathbf{x} + \hat{\mathbf{H}}^*\tilde{\mathbf{H}}\mathbf{x} + \hat{\mathbf{H}}^*\mathbf{n}\right] \end{aligned} \qquad (19)$$

and $E[\tilde{\mathbf{H}}\tilde{\mathbf{H}}^*] = M_T\mathcal{E}_{\tilde{h}}\mathbf{I}_{M_R}$. After some algebra, the matrix $\mathbf{R}_{ee,\mathrm{SM-HE}}$ can be written as

$$\mathbf{R}_{ee,\mathrm{SM-HE}} = E_s\left(\frac{1}{\rho} + \mathcal{E}_{\tilde{h}}\right)\left[\hat{\mathbf{H}}^*\hat{\mathbf{H}} + M_T\left(\mathcal{E}_{\tilde{h}} + \frac{1}{\rho}\right)\mathbf{I}_{M_T}\right]^{-1}. \qquad (20)$$

Hence,

$$\mathrm{SINR}_{i,\mathrm{SM-HE}} = \frac{1}{\left\{\left[\mathbf{I}_{M_T} + \frac{\rho}{M_T(1+\rho\mathcal{E}_{\tilde{h}})}\hat{\mathbf{H}}^*\hat{\mathbf{H}}\right]^{-1}\right\}_{i,i}} - 1. \qquad (21)$$

The mutual information for SM-HE is limited by the smallest post-processing SINR [1]. Thus, a lower bound on the mutual information for SM-HE conditioned on the channel estimate is given by

$$I_{\mathrm{SM-HE}} \geq M_T \log_2\left(1 + \min_{i\in\{1,\ldots,M_T\}} \mathrm{SINR}_{i,\mathrm{SM-HE}}\right). \qquad (22)$$

From (22), an upper bound on the outage probability for SM-HE is

$$\begin{aligned} &P_{\mathrm{out,SM-HE}} = \Pr[I_{\mathrm{SM-HE}} < R] \\ &\leq \Pr\left[\max_i\left\{\left[\mathbf{I}_{M_T} + \frac{\rho}{M_T(1+\rho\mathcal{E}_{\tilde{h}})}\hat{\mathbf{H}}^*\hat{\mathbf{H}}\right]^{-1}\right\}_{i,i} > 2^{-R/M_T}\right] \qquad (23) \\ &\leq \Pr\left[\lambda_{\min}\left(\frac{\hat{\mathbf{H}}^*\hat{\mathbf{H}}}{1-\mathcal{E}_{\tilde{h}}}\right) < \frac{M_T(1+\rho\mathcal{E}_{\tilde{h}})(2^{R/M_T}-1)}{\rho(1-\mathcal{E}_{\tilde{h}})}\right] \qquad (24) \end{aligned}$$

where (24) follows from Appendix A of [7] and $\lambda_{\min}(\cdot)$ denotes the minimum eigenvalue. Note that $\lambda_{\min}(\hat{\mathbf{H}}^*\hat{\mathbf{H}}/(1-\mathcal{E}_{\tilde{h}}))$ has the same distribution as $\lambda_{\min}(\mathbf{H}^*\mathbf{H})$. Hence, $F_{\lambda_{\min}}(\lambda) \leq F_Y(\lambda)$ where $F_{\lambda_{\min}}(\cdot)$ and $F_Y(\cdot)$ denote the cumulative distribution functions (cdfs) of $\lambda_{\min}(\hat{\mathbf{H}}^*\hat{\mathbf{H}}/(1-\mathcal{E}_{\tilde{h}}))$ and a random variable Y, respectively. Here, Y is a

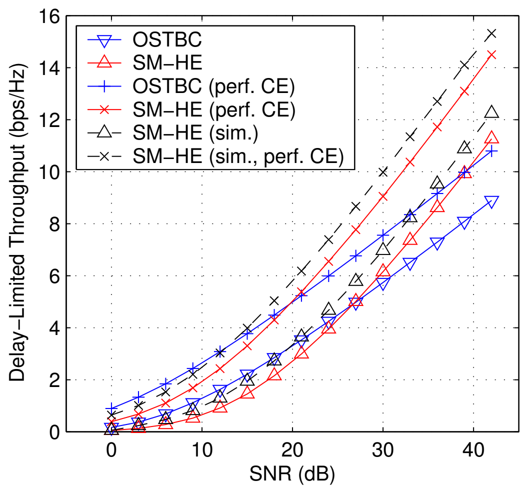

Fig. 1. Maximum lower bounds on delay-limited throughput versus SNR with channel estimation errors and $\alpha = 0.25$. Here, $M_T = 2$ and $M_R = 4$. Simulation results for SM-HE using (23) are labeled by "sim." Results with perfect channel estimation are labeled by "perf. CE."

gamma variable with parameters $(M_R - M_T + 1, 1/M_T)$ [8]. From the above relations, we have

$$P_{\mathrm{out,SM-HE}} \leq \frac{\gamma\left(M_R - M_T + 1, \frac{M_T^2(1+\rho\mathcal{E}_{\tilde{h}})(2^{R/M_T}-1)}{\rho(1-\mathcal{E}_{\tilde{h}})}\right)}{\Gamma(M_R - M_T + 1)}. \tag{25}$$

5. NUMERICAL RESULTS

In this section, the impact of channel estimation errors is evaluated using the lower bound on the delay-limited throughput given in (9) and the upper bounds on P_{out} given in (16), (23) and (25). From LMMSE channel estimation, $\mathcal{E}_{\tilde{h}}$ as a function of SNR is given by $\mathcal{E}_{\tilde{h}} = 1/(1+\alpha\rho)$ where α depends on the training time and the training SNR [4].

Fig. 1 is a plot of the lower bounds on the delay-limited throughput (maximized over R) versus SNR for OSTBC and SM-HE with $M_T = 2$ and $M_R = 4$. For OSTBC, the Alamouti code is used with $r_s = 1$. There exist efficient algorithms to determine the data rates R that maximize the throughput lower bounds. The target throughput outage level is $P_t = 0.1$, and the channel estimation error parameter is $\alpha = 0.25$. Throughput curves for perfect channel estimation are also included.

For these parameters, there is a loss of approximately 7 dB due to channel estimation errors. SM-HE throughput results are also plotted using simulations of the outage probability upper bound (23). There is a 2 dB gap between the SM-HE simulations and the throughput obtained from the analytical upper bound (25). It can be seen that OSTBC is preferred at low SNR, while SM-HE provides greater throughput at high SNR.

6. CONCLUSION

Delay-limited throughput is an important performance metric for real-time applications in quasi-static wireless channels. In this paper, lower bounds on the delay-limited throughput of OSTBC and SM-HE are computed to evaluate the impact of channel estimation errors. It is seen that channel estimation errors result in significant throughput loss for space-time coded systems. Spatial multiplexing yields higher throughput than OSTBC at high SNR as a result of the independent spatial data streams. At low SNR, OSTBC provides greater throughput because the increased diversity provides better robustness to fading and noise.

7. REFERENCES

[1] A. Paulraj, R. Nabar and D. Gore, *Introduction to Space-Time Wireless Communications*, Cambridge Univ. Press, 2003.

[2] S. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas in Commun.*, vol. 16, pp. 1451–1458, Oct. 1998.

[3] V. Tarokh, H. Jafarkhani and A. R. Calderbank, "Space-time block codes from orthogonal designs," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1456–1467, July 1999.

[4] J. Baltersee, G. Fock and H. Meyr, "Achievable rate of MIMO channels with data-aided channel estimation and perfect interleaving," *IEEE J. Selected Areas Commun.*, vol. 19, no. 12, pp. 2358–2368, Dec. 2001.

[5] E. Ko, C. Kang and D. Hong, "Effect of imperfect channel information on M-QAM SER performance of orthogonal space-time block codes," in *Proc. IEEE Vehic. Technol. Conf. Spring 2003*, Jeju, Korea, 21–24 Apr. 2003.

[6] M. Médard, "The effect upon channel capacity in wireless communications of perfect and imperfect knowledge of the channel," *IEEE Trans. Inform. Theory*, vol. 46, no. 3, pp. 933–946, May 2000.

[7] R. Narasimhan, "Spatial multiplexing with transmit antenna and constellation selection for correlated MIMO fading channels," *IEEE Trans. Signal Proc.*, vol. 51, no. 11, pp. 2829–2838, Nov. 2003.

[8] R. Narasimhan, A. Ekbal and J. M. Cioffi, "Finite-SNR diversity-multiplexing tradeoff of space-time codes," in *Proc. IEEE Intl. Conf. on Commun.*, Seoul, Korea, May 2005.