PERFORMANCE BREAKDOWN IN MUSIC, G-MUSIC AND MAXIMUM LIKELIHOOD ESTIMATION

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ABSTRACT

Performance of MUSIC and maximum likelihood direction-of-arrival estimation in the "threshold" region is compared with performance of the recently introduced G-MUSIC, based on the General Statistical Analysis (GSA) methodology. While the superiority of G-MUSIC over MUSIC has been demonstrated, it remains to be established whether G-MUSIC also outperforms MLE in scenarios within the threshold region. Comparisons of likelihood functions for MUSIC and G-MUSIC generated solutions as well as clairvoyantly optimized solutions are conducted to address this question.

Index Terms— Array signal processing, Maximum likelihood estimation, Adaptive estimation.

1. INTRODUCTION AND PROBLEM FORMULATION

In the absence of *a priori* distributions of the estimated parameters such as directions of arrival (DOAs) of multiple Gaussian signals impinging on an M-variate antenna array, the maximum likelihood (ML) criterion has been broadly treated as providing the "benchmark" estimation accuracy, if effectively implemented. Indeed, under asymptotic assumptions on the number of independent identically distributed (i.i.d) training samples T, the ML estimator (MLE) is proven to be asymptotically ($T \to \infty$) efficient, which means that its performance approaches the Cramér-Rao lower bound (CRLB) [1]. Moreover, it has been demonstrated by P. Stoica, et. al. in [2] that for independent Gaussian sources in noise, the well-known MUSIC estimation technique may be treated as a large-sample approximation of the ML estimator (MLE), with the same asymptotic accuracy. However, it has been known for a long time that when the sample support T (and/or signal-to-noise ratio) is insufficient, MUSIC performance "breaks down" and rapidly departs from the CRLB [3]. Typical manifestation of this breakdown is the appearance of severely erroneous DOA estimates ("outliers") that dramatically degrade the overall estimation accuracy. It is also recognized that the main phenomena which causes this breakdown is MUSIC-specific and is associated with the so-called "subspace-swap" [4,5].

But in addition to MUSIC, MLE also shows significant departure from the CRLB under certain conditions, even where the ML estimator could be accurately implemented [6]. Recently, we demonstrated in [7,8] that MLE suffers a performance breakdown at the point where solutions with completely erroneous DOA estimates (outliers) generate a likelihood function (LF) value that exceeds the LF values produced by the true solution, or even the local LF extremum of solutions with DOAs in the vicinity of the true solution.

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Investigations of the ML "performance breakdown" (threshold) conditions are ongoing [9,10], but it has already been demonstrated that there often is a significant "gap" in required sample support and/or SNR between MUSIC-specific and ML-intrinsic threshold conditions which may be practically addressed by ML-related routines [7,8]. Yet below the threshold region for ML, it is clear that maximization of the likelihood function (LF) is no longer associated with DOA estimation accuracy improvement and therefore the ML criterion is no longer adequate to the (DOA) estimation problem.

In this regard, it seems important to explore DOA estimation capabilities provided by new estimation approaches based on the novel "General Statistical Analysis" (GSA) methodology [11,12]. As with MLE, this methodology (known also as G-estimation) is also justified by asymptotic considerations, but there is a key distinction between the nature of asymptotic assumptions that support ML and G-estimation correspondingly. It is this distinction that makes the G-estimation paradigm more appropriate for applications in the "threshold" area with insufficient sample support.

Specifically, GSA addresses the familiar problem of estimation of some value $\varphi(R_M)$ [13], where φ is a continuous function of the entries of the covariance matrix R_M, represented by the training set of T independent identically distributed (i.i.d) M-variate observations (samples) $\mathbf{x}_1, \ldots, \mathbf{x}_T$.

Under traditional asymptotic assumptions when T is large and M is fixed

$$M = \text{constant}, \quad T \to \infty \tag{1}$$

and does not change when T grows, the estimator $\varphi(\hat{R}_M)$, where $\hat{R}_M$ is the sample (empirical) covariance matrix estimate $\hat{R}_M = \frac{1}{T}\sum_{j=1}^{T} x_j x_j^{\mathrm{H}}$, is consistent in the traditional sense:

$$\lim_{T\to\infty} \varphi(\hat{R}_M) = \varphi(R_M) \tag{2}$$

Moreover, for a Gaussian model, $\hat{R}_M$ is the ML covariance matrix estimate and therefore, according to theorem 5.1.1 in [14], $\varphi(\hat{R}_M)$ is also the MLE of $\varphi(R_M)$, which has been, in fact proven by P. Stoica et. al. explicitly for MUSIC [2].

However, assertion (2) is not valid (in general) in the asymptotic regime addressed by GSA, when M tends to infinity together with the number of observations T, i.e. the G-condition

$$\lim_{T\to\infty} MT^{-1} = \text{constant} < \infty \tag{3}$$

Under this G-condition, for a wide range of functions $\varphi(R_M)$, the GSA theory allows finding the limit (where limits of random quantities are understood to hold in some stochastic sense (almost surely or in probability):

$$\lim_{T\to\infty}[\varphi(\hat{R}_M) - \psi(R_M)] = 0, \tag{4}$$

where $\psi(R_M)$ is some measurable functions of entries of the matrix R_M. In general, the functions $\varphi(R_M)$ and $\psi(R_M)$ do not coincide, but with the help of the function $\psi(R_M)$, GSA theory provides the method of finding a measurable function $g(\hat{R}_M)$ such that

$$\lim_{T\to\infty}[g(\hat{R}_M) - \varphi(R_M)] = 0 \tag{5}$$

or the distribution of the normalized difference $g(\hat{R}_M) - \varphi R_M$ is asymptotically normal.

The function $g(\hat{R}_M)$ is called the G-estimator of $\varphi(R_M)$. The method of finding the G-estimator $g(\hat{R}_M)$ is quite complex, requiring tedious computer calculations. Yet, under some structural conditions, for the MUSIC pseudo-spectrum of m sources $\varphi(R_M, \theta_m)$,

$$\varphi(R_M, \theta_m) = S^{\mathrm{H}}(\theta) E_n E_n^{\mathrm{H}} S(\theta) \tag{6}$$

where E_n is the set of $(M-m)$ "noise subspace" eigenvectors of the covariance matrix, the G-estimator (G-MUSIC) has been derived by X. Mestre in [15, 16].

Since threshold DOA estimation conditions always consider finite M and T values and therefore a finite ratio $M/T = c$ value, for large enough M we may expect to be in a regime close to the G-condition (3), rather than the conventional asymptotic regime (1). Moreover, MUSIC-specific breakdown for strong signals is often observed in the so-called "under-sampled" regime, where the number of sources $m \ll M$, but still smaller than T. Here it is even more difficult to extrapolate from conventional asymptotic conditions (1).

Therefore, for sufficiently large M and finite ratio M/T, one can rightly expect that G-MUSIC should outperform conventional MUSIC, which is the large sample ML approximation. Indeed, the superiority of G-MUSIC over conventional MUSIC was recently reported by X. Mestre in [16], where he demonstrated that for a certain sample support T, "performance breakdown" in G-MUSIC takes place for significantly smaller SNR than in conventional MUSIC.

The remaining question is a similar comparison between the G-MUSIC and ML "performance breakdown" conditions. If the ML-intrinsic breakdown occurs in $M/T = c$ conditions where G-MUSIC does not produce outliers, then this fact may be treated as strong evidence of the ML paradigm decline. If on the contrary, the ML performance breakdown conditions are still more favorable than for G-MUSIC, then practical recommendations that follow from this comparison should favor ML-based breakdown "prediction and cure" techniques similar to the one suggested in [7, 17]. This is the question we address in this paper.

2. MUSIC, G-MUSIC AND ML DOA ESTIMATION: METHODOLOGY OF COMPARISON

Let $x_1, \ldots, x_T$ be the Gaussian i.i.d. training samples

$$X_j \sim \mathcal{CN}(0, R_M);\ \mathcal{E}\{x_j x_k^{\mathrm{H}}\} = \delta_{ij} R_M \tag{7}$$

where for the considered below scenario with m uncorrelated sources in additive white Gaussian noise (AWGN), the actual (true) covariance matrix R_M is specified as

$$R_M = \sum_{j=1}^{m} \sigma_j^2 S(\theta_j) S^{\mathrm{H}}(\theta_j) + \sigma_n^2 I \tag{8}$$

Here σ_j^2 and θ_j are the power and DOA of the j-th source, $S(\theta) \in C^{M\times 1}$ is the M-variate antenna manifold ("steering vector") in the direction θ and σ_n^2 is the AWGN power.

For simplicity, let us assume that the number of sources m and the noise power σ_n^2 are known *a priori* and therefore any estimator considered in this study generates the covariance matrix model:

$$R_{mod} = \left[\sum_{j=1}^{m} \sigma_{mod_j}^2 S(\theta_{mod_j}) S^{\mathrm{H}}(\theta_{mod_j})\right] + \sigma_n^2 I \tag{9}$$

where for a given set of DOA estimates $\hat{\theta}_1, \ldots \hat{\theta}_m$, the power estimates are found in the usual way [18]:

$$(\hat{\Sigma}) = diag_+ \left\{ [S^{\mathrm{H}}(\hat{\theta}) S(\hat{\theta})]^{-1} S^{\mathrm{H}}(\hat{\theta}) [\hat{R} - \sigma_0^2 I] S(\hat{\theta}) [S^{\mathrm{H}}(\hat{\theta}) S(\hat{\theta})]^{-1} \right\} \tag{10}$$

Here $diag_+\{A\}$ means the diagonal matrix build of positive diagonal entries of the matrix A, with negative entries a_{jj} replaced by zeros. One may consider more sophisticated power-matching techniques, but it is well known that ML matching is much less sensitive to the power errors than to the DOA errors.

Given the sample covariance matrix $\hat{R}$ with $T \geqslant m$, such that

$$\hat{R} = \hat{U}_m \hat{\Lambda}_m \hat{U}_m^{\mathrm{H}} + \hat{U}_n \hat{\Lambda}_n \hat{U}_n^{\mathrm{H}} \tag{11}$$

with $\hat{U}_j$ as the eigenvectors of the sample covariance matrix $\hat{R}$, with only $T-m$ non-zero noise-subspace eigenvalues in Λ_n for $T < M$, the MUSIC DOA estimates are found at the m greatest extrema of the MUSIC pseudo-spectra:

$$F_{MUSIC}(\theta) = \left[S^{\mathrm{H}}(\theta) \hat{U}_n \hat{U}_n^{\mathrm{H}} S(\theta)\right]^{-1}. \tag{12}$$

The derived in [16] G-MUSIC DOA estimates are specified by all the eigenvectors of $\hat{R}$:

$$F_G(\theta) = \left[S^{\mathrm{H}}(\theta) (\sum_{j=1}^{M} \phi_j \hat{U}_j \hat{U}_j^{\mathrm{H}}) S(\theta)\right]^{-1} \tag{13}$$

with the eigenvalues $\hat{\lambda}_j$, arranged in increasing order $\hat{\lambda}_1 \leqslant \hat{\lambda}_2 \leqslant \ldots \leqslant \hat{\lambda}_M$. For $T < M$, the first $M - T$ eigenvalues are equal to zero. For the known number m of point sources, the G-weighting function $\phi_j,\ j = 1, \ldots, M$ has been specified

$$\phi(j) = \begin{cases} 1 + \displaystyle\sum_{k=M-m+1}^{M} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_j - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_j - \hat{\mu}_k}\right),\ j \leqslant M-m \\ \quad - \displaystyle\sum_{k=1}^{M-m} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_j - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_j - \hat{\mu}_k}\right),\ j > M-m \end{cases} \tag{14}$$

with $\hat{\mu}_1 \leqslant \hat{\mu}_2 \leqslant \ldots \leqslant \hat{\mu}_M$ denoting the real valued solutions to the following equation in $\hat{\mu}$

$$\frac{1}{M} \sum_{k=1}^{M} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}} = \frac{T}{M} \tag{15}$$

Finally, the constant offset ϕ_0 is added to the weighting factors to keep them non-negative (i.e. $\phi_j + \phi_0 \geqslant 0$).

Based on MUSIC and G-MUSIC DOA estimations, two corresponding covariance matrix models R_{MUSIC} and R_G are constructed, as specified by (9), (10).

Therefore, we may now calculate the LF values $\mathcal{L}(X_T, R_{MUSIC})$ and $\mathcal{L}(X_T, R_G)$:

$$\mathcal{L}(X_T, R_{MUSIC}) = \left[\frac{\exp[-\mathrm{Tr}\, R_{MUSIC}^{-1}\hat{R}]}{\pi^M \det R_{MUSIC}}\right]^T \tag{16}$$

where $\mathcal{L}(X_T, R_G)$ is similarly introduced, and

$$\mathcal{L}(X_T, R_L) = \left[\frac{\exp[-\mathrm{Tr}\, R_L^{-1}\hat{R}]}{\pi^M \det R_L}\right]^T \tag{17}$$

where R_L is the result of a local LF maximization over the DOA set $\{\hat{\theta}_j\}$, performed in the vicinity of the true DOAs $\{\theta_j\}, j = 1, \ldots, m$. The accurate DOAs and powers are used to initialize this local optimization. The results of this optimization are used only to establish a local peak likelihood function value for thresholding (see (18)) and in our ML assessment that follows, we ignore the DoA estimates derived from this local optimization.

Since in the ML "threshold" area, this local extremum R_L *is not* necessarily the global one (in contrast to the behavior asymptotically in ML theory), strictly speaking, we need to perform the global search in order to find the globally optimal ML solution, in order to compare it with the G-MUSIC (and MUSIC) result.

For practically interesting scenarios with multiple sources impinging upon an antenna array, the global search is infeasible in most cases. Therefore, we consider a more pragmatic approach for our comparison. Specifically, in each trial, we compare

$$\begin{cases} \mathcal{L}(X_T, R_{MUSIC}) \geqslant \mathcal{L}(X_T, R_L) \\ \quad\quad \mathcal{L}(X_T, R_G) \geqslant \mathcal{L}(X_T, R_L) \end{cases} \tag{18}$$

Those trials that passed this inequality we correspondingly treat as the ones that belong to the set of admissible solutions (i.e. solutions that are as least as likely as the "proper" ML solution).

Since most of the MUSIC and G-MUSIC trials do not meet this inequality initially, we also perform refinement of R_{MUSIC} and R_G models that fail the threshold, in order to get solutions R_{MR} (MUSIC refined) and R_{GR} (G-MUSIC refined) that pass these inequalities.

Obviously, since our refinement routine is not the global search, the success rate of such ML optimizations is less than ideal, which means that in some trials, we do not achieve the specified LF values. Yet the assessment of ML accuracy can be performed over the set of trials that do pass this clairvoyantly derived inequality, limiting our ML performance assessment to those trials where performance similar to that of a global maximization is achieved.

The refinement routine mentioned above follows our practical rectification routine mentioned in detail in [17, 18]. Specifically, among the set of DOA estimates $\{\hat{\theta}_j\}$ we first specify an "outlier" as the estimate least contributing to the LF value. In result of a 1-D LF search, we find a DOA estimate that substitutes an "outlier", and then perform local LF optimization in the vicinity of the new DOA set. If necessary, we repeat the entire procedure, until the required LF value is achieved, or the number of iterations exceeds a pre-defined value. The main distinction between the "practical" routines introduced in [17, 18] and this "impractical" routine are the threshold LF values. Within our "practical" routines, we introduce some properly normalized likelihood ratios instead of the likelihood function, and use the invariance property of these LR p.d.f.s for $R_{mod} = R_0$. This invariance allows us to replace the "strict" threshold (18) by a scenario-invariant statistical one, that must be exceeded by the $LR(R_0)$ with a certain (high) probability. Because of the similarity to the generalized likelihood-ratio test (GLRT) used in adaptive detection, we refer to this GLRT-based "prediction and cure" technique in "practical" circumstances as GLRT-PAC. Correspondingly, the "impractical" routine used here is referred to at LF-PAC.

In this study, we use the clairvoyantly known value $\mathcal{L}(X_T, R_L)$ within a LF-PAC framework in order to explore potential capabilities of the ML methodology.

3. SIMULATION RESULTS

Our goal is to compare MUSIC, G-MUSIC and MLE performance in the threshold area that spans the range from "proper" MUSIC behavior (no outliers) up to the G-MUSIC and ML complete "performance breakdown". For this reason, we once again consider the scenario already addressed in [15], with a ($M = 20$)-element uniform linear array (ULA), element spacing of $d/\lambda = 0.5$ and $m = 4$ independent Gaussian sources immersed in white noise.

The following DOA set was considered:

$$\theta_m = \{-20^o, -10^o, 35^o, 37^o\} \tag{19}$$

with equal SNR for each source, varying within the range from -15dB to 25dB, while the sample support was selected, as in [15], equal to $T = 15$ (i.e. $M/T = 1.33\ldots$)

Fig. 1 shows the mean square error (averaged over 30 trials) for DOA estimates of the two closely spaced sources (at 35^o and 37^o). Also shown is the stochastic Cramér-Rao lower bound (CRLB) averaged over the sources with DOA 35^o and 37^o. The figures were generated in a manner similar to Figure 4 in [15], but with a finer SNR spacing to show more clearly the onset of the "threshold effect" of outlier production. In all cases, the number of sources was considered known *a priori*, with $m = 4$. First one can see that within the entire SNR domain shown (-15 to 25 dB), the actual DOA estimation accuracy of G-MUSIC remains superior to the MUSIC estimation accuracy.

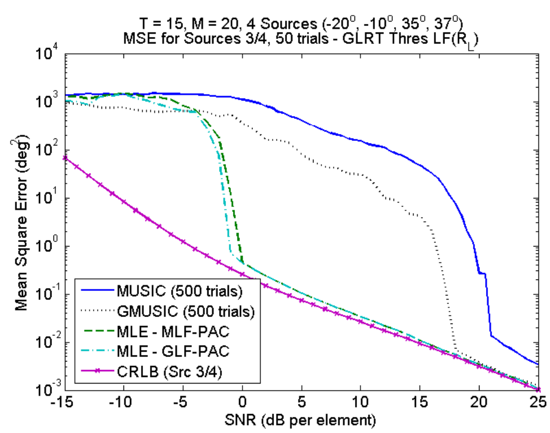

Fig. 1: MSE for MUSIC, G-MUSIC and MLE (via LF-PAC)

The breakpoint where the accuracy starts to rapidly depart from the CRLB due to the first outliers is around 17dB for G-MUSIC and

20dB for MUSIC, as was already reported for this scenario in [16] (after a SNR grid spacing modification discussed in a personal communication [19]). Everywhere beyond this point (SNR = 17dB), the actual performance of G-MUSIC and MUSIC is far worse than the CRLB. The fact that in the threshold area, the CRLB dramatically under-estimates the actual accuracy was perfectly known for MUSIC, and now is demonstrated for G-MUSIC as well.

The percentage of trials that contain an estimation error greater than 2^o degrees (an outlier) is illustrated for G-MUSIC and MUSIC at Fig. 2 for varying SNR. This comparison makes clear that MUSIC deteriorates much more rapidly than G-MUSIC, reinforcing the conclusion on essential G-MUSIC superiority over the conventional MUSIC in the threshold area made in [16].

Yet examination of the MSE curves in Fig. 1 and the outlier production rate curves in Fig. 2 for the MLE methodology discussed above show a large shift in the onset of the threshold area, from 17-20dB down to 3dB for both our MUSIC and G-MUSIC "seeded" ML optimization routines (labelled GLF-PAC and MLF-PAC for G-MUSIC initiated and MUSIC initiated LF optimization prediction and cure). As expected for a technique designed to converge on the MLE solution, the initial estimate source (G-MUSIC or MUSIC) makes little difference in the performance of the ML-based post-processing technique. As further confirmation that the LF-PAC routine is providing a reasonable estimate of global MLE performance, the "statistical resolution limit" [20] (where the source separation is equal to the CRLB of each source) coincides with the point where outlier production dominates over proper solutions.

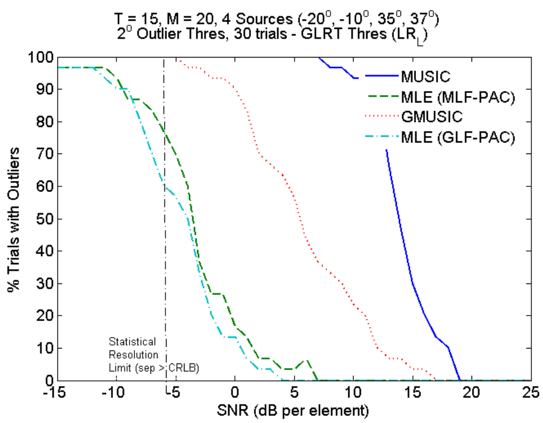

Fig. 2: % Outliers for Standard and LF-PAC aided MUSIC/G-MUSIC

4. SUMMARY AND CONCLUSION

Comparative performance analysis of the MUSIC, G-MUSIC, and MLE DOA estimation accuracy demonstrates that in the "threshold" area, G-MUSIC significantly outperforms conventional MUSIC, but its performance still remains inferior to the potential performance of maximum likelihood estimation. This is despite the choice of a small sample size scenario which favors the asymptotic assumptions associated with the derivation of G-MUSIC. Specifically, for the considered scenario, we demonstrated that the SNR "performance breakdown" (threshold) values are equal to 20 dB, 17dB, and 3dB for MUSIC, G-MUSIC, and MLE correspondingly. The MLE proxy (LF-PAC) used in analysis was based on clairvoyant knowledge of the solution, but practical implementations based on the same technique are also available and are a computationally efficient ML-based technique in the threshold region [21].

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