RELATIVE AND ABSOLUTE ERRORS IN SENSOR NETWORK LOCALIZATION

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ABSTRACT

This paper considers the accuracy of sensor node location estimates from self-calibration in sensor networks. The location parameters are shown to have a natural decomposition into relative configuration and centroid transformation components based on the influence of measurements and prior information in the problem. A linear representation of the transformation parameter space, which includes rotations and translations, is shown to coincide with the nullspace of the unconstrained Fisher information matrix (FIM). To regularize the absolute localization problem, we consider constraints on the coordinate locations and the impact of these constraints on relative and absolute location error. A geometric interpretation of the constrained Cramér-Rao bound (CRB) is provided based on the principal angles between the measurement subspace and the constraint subspace. Examples illustrate the utility of this error decomposition.

Index Terms— Sensor networks, Localization, Constrained estimation, Cramér-Rao bound, Fisher information

1. INTRODUCTION

In a distributed wireless sensor network, knowledge of the sensor locations is a prerequisite to obtaining meaningful information from measurements made by the sensors. As such, a diverse variety of self-localization algorithms based on some form of inter-node measurements have been proposed in the literature. In order to better understand how noise, deployment geometry, and measurement type effect fundamental location estimation performance, a number of authors have considered the Cramér-Rao bound (CRB) on self-localization performance (see eg. [1, 2] and references therein). In this paper we extend the general CRB analysis by providing a meaningful decomposition of localization error.

In particular, we decompose the total localization error into a relative portion representing error in the estimated network shape and a transformation portion representing error in the absolute position of the relative scene. This decomposition is motivated by the fact that relative information is derived from both measurements and prior information, while transformation information comes solely from prior information. Because the inter-node calibration measurements provide no information about the transformation parameters, we regularize the problem by considering general parametric constraints on the sensor locations.

One of the main contributions of this work is an analysis illustrating how the constraint subspace interacts with the measurement subspace to effect total localization performance. Along with the CRB itself, the relative / transformation decomposition presented here gives insight into how external inputs effect absolute localization. This partitioning of error is also useful to higher level applications in a sensor network that utilize results of the localization service.

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2. RELATIVE AND TRANSFORMATION ERROR

2.1. Formulation

The absolute self-localization problem is to combine inter-node measurements collected in a measurement vector $\boldsymbol{z}$ with prior information in order to obtain estimates of the coordinates $\{p_i = [x_i \ y_i]^T\}_{i=1}^N$ of the N constituent nodes of the network. A general measurement model takes the form

$$\boldsymbol{z} = \boldsymbol{\mu}(\boldsymbol{\theta}) + \boldsymbol{\eta} \in \mathbb{R}^M, \tag{1}$$

where $\boldsymbol{z}$ is the vector of M measurements, $\boldsymbol{\mu}$ is the mean of the observation which is structured by the true coordinate parameter vector $\boldsymbol{\theta} = [x_1 \, y_1 \ \ldots \ x_N \, y_N]^T$, and $\boldsymbol{\eta}$ is a zero-mean noise vector. In this paper we consider inter-node distance measurements, hence elements of $\boldsymbol{\mu}$ are of the form $||p_i - p_j||$. Since inter-node distances are invariant to the scene's absolute position and orientation, the measurements only inform upon the relative shape of the network. As observed in [3], this manifests itself as a singular Fisher information matrix, J

$$J \quad = \quad [U_J \widetilde{U}_J] \begin{bmatrix} \Lambda_J & 0 \\ 0 & 0 \end{bmatrix} [U_J \widetilde{U}_J]^T, \tag{2}$$

whose nullspace $\mathcal{R}(\widetilde{U}_J)$ is spanned by the vectors

$$\boldsymbol{v}_x = \alpha_x \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \boldsymbol{v}_y = \alpha_y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \boldsymbol{v}_\phi = \alpha_\phi \begin{bmatrix} -(y_1 - \bar{y}) \\ (x_1 - \bar{x}) \\ -(y_2 - \bar{y}) \\ (x_2 - \bar{x}) \\ \vdots \end{bmatrix}, \tag{3}$$

where the scalers α_x, α_y, and α_ϕ are chosen such that the vectors in (3) have unit norm. The vectors $\boldsymbol{v}_x$ and $\boldsymbol{v}_y$ represent translations in the x- and y-directions, respectively, and $\boldsymbol{v}_\phi$ represents a linearization of the rotation operation of each point $\boldsymbol{p}_i$ about $(\bar{x}, \bar{y})$, where $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$ and $\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ are the x and y centroids.

Absolute localization based on measurements alone is inherently ill-posed as the translation and rotation components of $\boldsymbol{\theta}$ in $\mathcal{R}(\bar{U}_J)$ cannot be estimated. To regularize the problem, we consider generic parametric constraints of the form

$$\boldsymbol{f}(\boldsymbol{\theta}) = \boldsymbol{0}. \tag{4}$$

In general, $\boldsymbol{f}(\boldsymbol{\theta})$ is a k-vector representing a system of k constraints. For example, to constrain the centroid ($k = 2$) to the origin $(0, 0)$, we have

$$\boldsymbol{f}(\boldsymbol{\theta}) = \frac{1}{N}[\boldsymbol{v}_x\ \boldsymbol{v}_y]^T\boldsymbol{\theta}. \tag{5}$$

This constraint formulation represents a generalization of the typical use of anchor (also called beacon) nodes.

From [4], the CRB for an unbiased estimator $\hat{\boldsymbol{\theta}}$ satisfying a constraint $\boldsymbol{f}(\hat{\boldsymbol{\theta}}) = \boldsymbol{0}$ is

$$E[(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T] \geq U_c(U_c^T J U_c)^{-1}U_c^T \triangleq \Sigma_c, \tag{6}$$

where U_c is a semiunitary matrix whose columns form an orthonormal basis for the nullspace of the gradient matrix $F = \frac{\partial \boldsymbol{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}^T} \in \mathbb{R}^{k\times 2N}$. We assume that the constraint $\boldsymbol{f}(\boldsymbol{\theta})$ is chosen such that the inverse in (6) exists.

2.2. Error decomposition

Let $\hat{\boldsymbol{\theta}} = [\hat{x}_1\ \hat{y}_1\ \ldots\ \hat{x}_N\ \hat{y}_N]^T$ denote the parameter estimate given by some estimator. If the estimator did not yield the optimal transformation parameters (translation and rotation), then the error

$$\epsilon = ||\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}||^2, \tag{7}$$

can be further reduced by applying a planar transformation to the estimates $\hat{\boldsymbol{\theta}}$. Note that $\epsilon = \sum_{i=1}^{N} d_i^2$, where d_i is the distance between node i and its estimate. We denote the optimally translated and rotated estimates as $\hat{\boldsymbol{\theta}}_r$, which may be approximated by $\hat{\boldsymbol{\theta}}_r \approx \tilde{\boldsymbol{\theta}}_r = \hat{\boldsymbol{\theta}} + W\hat{\boldsymbol{\beta}}$, where $W = [\boldsymbol{v}_x\ \boldsymbol{v}_y\ \boldsymbol{v}_\phi]$, and the optimal transformation coefficients are

$$\begin{aligned} \hat{\boldsymbol{\beta}} &\triangleq \arg\min_{\boldsymbol{\beta}} ||\hat{\boldsymbol{\theta}} + W\boldsymbol{\beta} - \boldsymbol{\theta}||^2 && (8)\\ &= W^T(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) && (9)\\ &= [\hat{\boldsymbol{\beta}}_x\ \hat{\boldsymbol{\beta}}_y\ \hat{\boldsymbol{\beta}}_\phi]^T. && (10) \end{aligned}$$

As the translation and rotation components of $\hat{\boldsymbol{\theta}}$ have been corrected in $\tilde{\boldsymbol{\theta}}_r$, the error $\epsilon_r \triangleq ||\boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}_r||^2$ represents the relative error, or the error in the "shape" of $\hat{\boldsymbol{\theta}}$. We define the transformation error, ϵ_t, as the portion of the total error due to miss-estimation of the transformation parameters, $\epsilon_t \triangleq \epsilon - \epsilon_r$. As illustrated in Figure 1, ϵ_r and ϵ_t are easily represented in terms of $\tilde{\boldsymbol{w}}_t$ and $\tilde{\boldsymbol{w}}_r$, the projections of the error vector $\boldsymbol{\xi} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ onto $\mathcal{R}(W)$ and $\mathcal{R}(W)^\perp$, respectively.

For an unbiased estimator, we may express the expected values of ϵ, ϵ_r and ϵ_t in terms of the estimator covariance matrix $\Sigma_{\hat{\boldsymbol{\theta}}} = E[\boldsymbol{\xi}\boldsymbol{\xi}^T]$. Let $\Sigma_t = E[\boldsymbol{\beta}\hat{\boldsymbol{\beta}}] = W^T\Sigma_{\hat{\boldsymbol{\theta}}}W$ denote the covariance matrix of the transformation coefficients, and let $\Sigma_r = E[\tilde{\boldsymbol{w}}_r\tilde{\boldsymbol{w}}_r^T] = (\tilde{W}\tilde{W}^T)\Sigma_{\hat{\boldsymbol{\theta}}}(\tilde{W}\tilde{W}^T)$ denote the covariance matrix of the error in the relative subspace $\mathcal{R}(W)^\perp$, where the columns of $\tilde{W}$ form an orthonormal basis for $\mathcal{R}(W)^\perp$. The expected errors are

$$\begin{aligned} e &\triangleq E[\epsilon] = E[\boldsymbol{\xi}^T\boldsymbol{\xi}] = \mathrm{tr}\ \Sigma_{\hat{\boldsymbol{\theta}}} && (11)\\ e_r &\triangleq E[\epsilon_r] = E[\tilde{\boldsymbol{w}}_r^T\tilde{\boldsymbol{w}}_r] = \mathrm{tr}\ \Sigma_r && (12)\\ e_t &\triangleq E[\epsilon_t] = E[\tilde{\boldsymbol{w}}_t^T\tilde{\boldsymbol{w}}_t] = \mathrm{tr}\ \Sigma_t, && (13) \end{aligned}$$

and, as desired, the sum of the mean component errors equals the total: $e = \mathrm{tr}\ [\tilde{W}W]\ \Sigma_{\hat{\boldsymbol{\theta}}}\ [\tilde{W}W]^T = e_r + e_t$.

For a given constraint function $\boldsymbol{f}(\boldsymbol{\theta})$ and noise distribution $p_\eta(\boldsymbol{\eta})$, lower bounds on the expected errors e, e_r and e_t may be obtained by substituting the constrained CRB Σ_c in (6) for $\Sigma_{\hat{\theta}}$ in (11), (12), and (13).

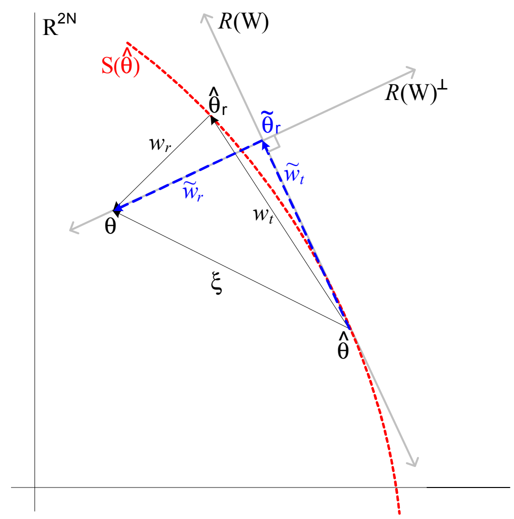

Fig. 1: Geometric illustration of relative and transformation errors in the location parameter vector $\boldsymbol{\theta}$. The manifold $S(\hat{\boldsymbol{\theta}})$ represents rigid translations and rotations of the coordinate estimates $\hat{\boldsymbol{\theta}}$, and the point on $S(\hat{\boldsymbol{\theta}})$ closest to $\boldsymbol{\theta}$ represents the optimally transformed estimate, $\hat{\boldsymbol{\theta}}_r$. The error vector $\boldsymbol{\xi} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$ may be decomposed into $\boldsymbol{\xi} = \boldsymbol{w}_r + \boldsymbol{w}_t$, where $\boldsymbol{w}_r = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_r$ is the relative error vector and $\boldsymbol{w}_t = \hat{\boldsymbol{\theta}}_r - \hat{\boldsymbol{\theta}}$. $\boldsymbol{w}_t$ and $\boldsymbol{w}_r$ may be approximated, respectively, by $\tilde{\boldsymbol{w}}_t$ and $\tilde{\boldsymbol{w}}_r$, the projections of the error vector $\boldsymbol{\xi}$ onto the transformation subspace $\mathcal{R}(W)$ and the relative subspace $\mathcal{R}(W)^\perp$.

3. GEOMETRIC INTERPRETATIONS

In this section we provide a geometric interpretation of how the total error, $\mathrm{tr}\ \Sigma_c$, depends on J and properties relating the subspaces $\mathcal{R}(U_J)$ and $\mathcal{R}(U_c)$. Let $A = U_J^T U_c$, and consider its SVD

$$A = [Y_1 Y_2] \begin{bmatrix} \Lambda_A \\ 0 \end{bmatrix} Z^T, \tag{14}$$

where $Y_1 \in \mathbb{R}^{2N-3 \times 2N-k}$ and $Y_2 \in \mathbb{R}^{2N-3 \times k-3}$. The singular values, $\sigma_1(A) \geq \cdots \geq \sigma_{2N-k}(A)$, correspond to the cosines of the principal angles, $0 \leq \phi_1 \leq \cdots \leq \phi_{2N-k} \leq \pi/2$, between the subspaces $\mathcal{R}(U_c)$ and $\mathcal{R}(U_J)$, [5, Ch. 12]

$$\cos \phi_i = \sigma_i(A). \tag{15}$$

It can be shown that [6]

$$\sum_{i=1}^{2N-k} \frac{1}{\cos^2 \phi_i} \frac{1}{\sigma_{2N-k+1-i}(J)} \leq \operatorname{tr} \Sigma_c \leq \sum_{i=1}^{2N-k} \frac{1}{\cos^2 \phi_i} \frac{1}{\sigma_{k-d+i}(J)}. \tag{16}$$

The lower bound consists of a weighted sum of the reciprocals of the largest $(2N-k)$ singular values, and the upper bound uses the reciprocals of the $(2N-k)$ smallest non-zero singular values of J. The weightings depend on the principal angles between $\mathcal{R}(U_c)$ and $\mathcal{R}(U_J)$.

Examining (16) we can see how the interplay between the two information sources – constraints and measurements – influences the total estimation error. Considering a linearization of the constraint $\boldsymbol{f}(\hat{\boldsymbol{\theta}})$ about $\boldsymbol{\theta}$,

$$\boldsymbol{f}(\hat{\boldsymbol{\theta}}) \approx \boldsymbol{f}(\boldsymbol{\theta}) + F\,(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}) = \mathbf{0} \tag{17}$$

$$\implies F\hat{\boldsymbol{\theta}} = F\boldsymbol{\theta} - \boldsymbol{f}(\boldsymbol{\theta}), \tag{18}$$

we see that, for $\hat{\boldsymbol{\theta}} \approx \boldsymbol{\theta}$, the constraint function precisely determines $\hat{\boldsymbol{\theta}}$ in the k-dimensional subspace $\mathcal{R}(F^T)$ but says nothing about the components of $\hat{\boldsymbol{\theta}}$ in $\mathcal{R}(F^T)^\perp = \mathcal{R}(U_c)$. We call $\mathcal{R}(F^T)$ and $\mathcal{R}(U_c)$ the constrained and unconstrained subspaces under constraint $\boldsymbol{f}$, respectively.

The parameter space $\mathbb{R}^{2N}$ may also be partitioned from the measurements into $\mathcal{R}(U_J)$ and $\mathcal{R}(\widetilde{U}_J)$, where $\mathcal{R}(U_J)$ represents the subspace informed upon by measurements, and $\mathcal{R}(\widetilde{U}_J) = \mathcal{R}(W)$ represents the transformation subspace which is not estimable from measurements.

When the unconstrained subspace, $\mathcal{R}(U_c)$ is closely aligned with the measurement subspace, $\mathcal{R}(U_J)$, the principal angles are small and the estimation performance, from (16), is good.

If, for a minimally constrained system, the $k = 3$ constraints precisely determine the components of $\hat{\boldsymbol{\theta}}$ in $\mathcal{R}(\widetilde{U}_J)$, then $\mathcal{R}(U_c) = \mathcal{R}(U_J)$ and we may write $U_c = U_J B$, for some non-singular matrix B. As $\mathcal{R}(\widetilde{U}_J) = \mathcal{R}(W)$, this corresponds to a constraint that fully specifies the unknown transformation parameters; that is, the scene centroid and orientation are fully specified. In this case, the CRB from (6) may be rewritten as

$$\begin{aligned} \Sigma_c &= U_c (B^T \Lambda_J B)^{-1} U_c^T \\ &= U_J \Lambda_J^{-1} U_J^T = J^\dagger. \end{aligned} \tag{19}$$

In the localization context, the pseudo-inverse $J^\dagger$ was considered in [3] and referred to as the relative CRB, and later in [7] being called the anchor-free CRB. This paper generalizes the relative CRB concept to the case of general constraints, and provides a geometric understanding of the subspaces involved.

4. EXAMPLE: ANCHOR SELECTION

Anchor nodes, that is, nodes with *a priori* known location coordinates, provide a particular type of the more general constraint (4). Let $\mathcal{A}$ denote a set of anchor nodes with coordinates collected in the vector $\boldsymbol{\theta}_\mathcal{A}$. Then, for anchors, the constraint function takes the linear form

$$\boldsymbol{f}(\boldsymbol{\theta}) = H\,\boldsymbol{\theta} - \boldsymbol{\theta}_\mathcal{A} = \mathbf{0}, \tag{20}$$

where the rows of H correspond to appropriate columns of the identity matrix in order to extract the known coordinates from $\boldsymbol{\theta}$. We consider the selection of three anchor nodes in order to localize, with an absolute reference, the network in Figure 2. All node pairs make distance measurements with Gaussian measurement error $\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, \sigma_m^2 I)$, where $\sigma_m = 2\,\text{m}$. For anchors $\mathcal{A} = \{9, 13, 14\}$, the figure illustrates 2-σ uncertainty ellipses for each node corresponding to the total constrained CRB Σ_c, and the relative portion Σ_r.

From Figure 2 we observe a large radial uncertainty in the total error because the anchors are clustered at one corner of the network. This radial "smearing" is much less pronounced in Σ_r which eliminates uncertainty in the transformation parameters. The total estimation error may be improved by selecting better anchor points. Two heuristic selection mechanisms are to choose anchors that 1) cover a maximal area, or 2) have a maximal perimeter. In Figure 3 we plot the transformation error e_t, and the relative error e_r, as a function of all possible $\binom{16}{3}$ anchor node triples, sorted by decreasing e_t. From the figure we see that different anchor sets have little effect on the relative error but have a significant effect on the transformation error. The total absolute localization error is the sum of the relative and transformation portions. The optimal anchor set with the minimum total error is $\mathcal{A} = \{1, 4, 13\}$ and is illustrated by the vertical bar in Figure 3. For this case, the maximum perimeter heuristic yielded the optimal anchor set, and the maximum area heuristic yielded $\mathcal{A} = \{1, 8, 13\}$ with an error only 0.5% greater than the optimal error. In general, neither heuristic gives the lowest total error, but both result in localization estimates very close to optimal for a large number of example networks that have been considered.

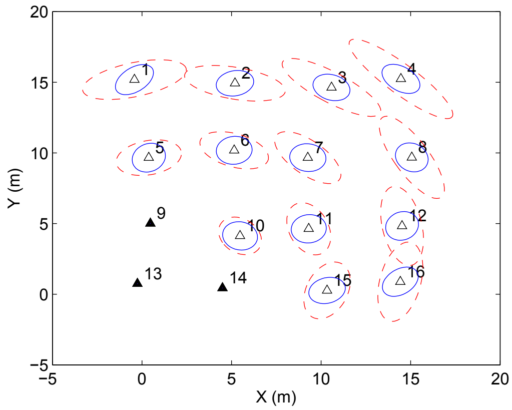

Fig. 2: Sensor positions and associated 2-σ uncertainty ellipses corresponding to the total constrained CRB Σ_c (- -), and the relative bound Σ_r (—). The total error exhibits large radial uncertainty which is not seen in the relative subspace. The anchor set was $\mathcal{A} = \{9, 13, 14\}$ for both cases.

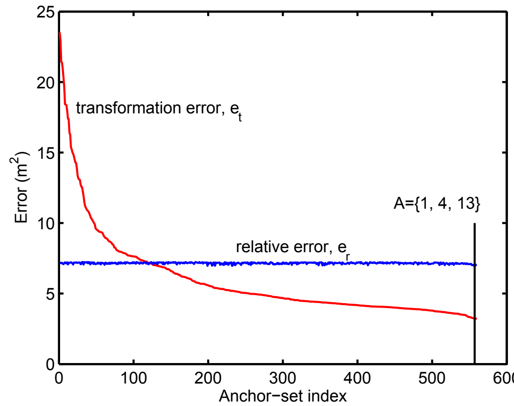

Fig. 3: The transformation error e_t and relative error e_r exhibit very different responses to alterations in the constraint function — which is achieved by selecting different anchor triplets of the network in Figure 2. The total error is the sum of these two curves, $e = e_t + e_r$, and is minimized by the anchor set $\mathcal{A} = \{1, 4, 13\}$.

5. CONCLUSIONS

Node localization error has a natural decomposition into relative and transformation components, due to the nature of localization measurements and the influence of prior constraint information. Transformation error represents error in the translation and rotation of a relative solution and is only informed upon by prior constraints. Relative error represents error in the estimated "shape" and is derived from both measurements and constraints. The decomposition is readily extended to other measurement types including time-difference and arrival-angle measurements [6].

In this paper we presented general results (not specific to localization) that relate bounds on total estimation error in a constrained system to the unconstrained Fisher information matrix and the principal angles between the measurement subspace $\mathcal{R}(U_J)$ and the unconstrained subspace $\mathcal{R}(U_c)$.

The results of this paper provide insight into how different information sources impact different parts of the final localization estimate. There are three general areas which can benefit from the results in this paper. The first area involves relative-only estimators, such as Isomap [8] and multidimensional scaling. In this case, the relative CRB (19) is the appropriate benchmark. The second area includes development of new absolute localization algorithms whose performance and design may be analyzed with respect to relative and transformation components separately. The example in Section 4 illustrated how these components behaved very differently with respect to anchor selection. The final area includes applications relying on localization results where it may be beneficial to dissect a localization error covariance matrix into transformation error Σ_t, and relative error Σ_r. For example, a source tracking algorithm may initially obtain a track relative to the sensors, and subsequently apply Σ_t to prescribe translation and rotation variability to the estimated track.

6. REFERENCES

[1] N. Patwari, J. Ash, S. Kyperountas, A. Hero III, R. Moses, and N. Correal, "Locating the nodes: cooperative localization in wireless sensor networks," *IEEE Signal Processing Mag.*, vol. 22, no. 4, pp. 54–69, July 2005.

[2] A. Savvides, W. Garber, R. Moses, and M. Srivastava, "An analysis of error inducing parameters in multihop sensor node localization," *IEEE Trans. Mobile Comput.*, vol. 4, no. 6, pp. 567–577, 2005.

[3] R. Moses and R. Patterson, "Self-calibration of sensor networks," *Unattended ground sensor technologies and applications IV, Proc. SPIE vol. 4743*, pp. 108–119, 2002.

[4] P. Stoica and B. Ng, "On the Cramér-Rao bound under parametric constraints," *IEEE Signal Processing Lett.*, vol. 5, no. 7, pp. 177–179, 1998.

[5] G. Golub and C. Van Loan, *Matrix Computations*, Johns Hopkins University Press, 3^{rd} edition, 1996.

[6] J. Ash and R. Moses, "Decomposition of transformation and relative positioning error in self-localization systems," *IEEE Trans. Signal Processing*, 2006 (in submission).

[7] C. Chang and A. Sahai, "Cramér-Rao type bounds for localization," *EURASIP Journal on Applied Signal Processing*, vol. 2006, 2006.

[8] J. Tenenbaum, V. de Silva, and J. Langford, "A global geometric framework for nonlinear dimensionality reduction," *Science*, vol. 290, no. 5500, pp. 2319–2323, 2000.