# WAVEFORM OPTIMIZATION FOR MIMO RADAR: A CRAMÉR-RAO BOUND BASED STUDY

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## ABSTRACT

A MIMO (multi-input multi-output) radar system, unlike standard phased-array radar, can transmit via its antennas multiple probing signals. This waveform diversity offered by MIMO radar enables superior capabilities compared with a standard phased-array radar. We consider MIMO radar waveform optimization for parameter estimation for the general case of multiple targets in the presence of spatially colored interference and noise. Numerical examples are provided to demonstrate the effectiveness of the approaches we consider herein.

***Index Terms*— Multiple-Input Multiple-Output (MIMO) Radar, Cramér-Rao Bound (CRB), Waveform optimization**

## I. INTRODUCTION

MIMO (multi-input multi-output) radar is an emerging technology that is attracting the attention of researchers and practitioners alike. Unlike a standard phased-array radar, which transmits scaled versions of a single waveform, a MIMO radar system can transmit via its antennas multiple probing signals that may be quite different from each other. This waveform diversity offered by MIMO radar enables superior capabilities compared with a standard phased-array radar; see, e.g., [1] - [8] and reference therein.

In this paper, we consider MIMO radar waveform optimization for parameter estimation for the general case of multiple targets in the presence of spatially colored interference and noise. Some of our results can be seen as significant extensions of those presented in [5], where the parameter estimation of a single target in the presence of spatially and temporally white noise is considered.

## II. PROBLEM FORMULATION

Consider a MIMO radar equipped with co-located antennas. Let $N$ and $M$, respectively, denote the numbers of transmit and receive antennas. The data matrix received by such a MIMO radar can be written as:

$$\mathbf{X} = \sum_{k=1}^{K} \mathbf{a}(\theta_k) b_k \mathbf{v}^T(\theta_k)\mathbf{\Phi} + \mathbf{Z}, \tag{1}$$

where the columns of $\mathbf{X} \in C^{M\times L}$ are the received data vectors, with $L$ being the snapshot number; $\{\theta_k\}_{k=1}^K$ are the locations of the targets with $K$ being the number of targets at a particular range bin of interest; $\mathbf{a}(\theta) \in C^{M\times 1}$ and $\mathbf{v}(\theta) \in C^{N\times 1}$ are the steering vectors for the receiving and transmitting arrays, respectively; $\{b_k\}_{k=1}^K$ are the target complex amplitudes, which are proportional to the radar-cross-sections (RCS) of the targets; the rows of $\mathbf{\Phi} \in C^{N\times L}$ are the transmitted waveforms, which are known and deterministic; $\mathbf{Z}$ is the interference and noise term, which includes the responses due to targets at other range bins; and $(\cdot)^T$ denotes the transpose. We assume that the columns of $\mathbf{Z}$ are independent and identically distributed circularly symmetric complex Gaussian random vectors with mean zero and an unknown covariance matrix $\mathbf{Q}$.

For notational simplicity, (1) can be rewritten as:

$$\mathbf{X} = \mathbf{A}(\boldsymbol{\theta})\mathbf{B}\mathbf{V}^T(\boldsymbol{\theta})\mathbf{\Phi} + \mathbf{Z}, \tag{2}$$

where

$$\mathbf{A} = [\mathbf{a}(\theta_1)\ \cdots\ \mathbf{a}(\theta_K)], \quad \mathbf{V} = [\mathbf{v}(\theta_1)\ \cdots\ \mathbf{v}(\theta_K)], \tag{3}$$

$$\boldsymbol{\theta} = [\theta_1\ \cdots\ \theta_K]^T,\ \mathbf{b} = [b_1\ \cdots\ b_K]^T,\ \mathbf{B} = \mathrm{diag}(\mathbf{b}), \tag{4}$$

with $\mathrm{diag}(\mathbf{b})$ denoting a diagonal matrix with $\mathbf{b}$ being its diagonal.

## III. CRAMÉR-RAO BOUND

The Cramér-Rao bound matrix for the unknown target parameters $\boldsymbol{\theta}$, $\mathrm{Re}(\mathbf{b})$ and $\mathrm{Im}(\mathbf{b})$ can be written as follows (see [9] for the derivation):

$$\mathbf{C} = \mathbf{F}^{-1}, \tag{5}$$

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where

$$\mathbf{F} = 2\begin{bmatrix} \mathrm{Re}(\mathbf{F}_{11}) & \mathrm{Re}(\mathbf{F}_{12}) & -\mathrm{Im}(\mathbf{F}_{12}) \\ \mathrm{Re}^T(\mathbf{F}_{12}) & \mathrm{Re}(\mathbf{F}_{22}) & -\mathrm{Im}(\mathbf{F}_{22}) \\ -\mathrm{Im}^T(\mathbf{F}_{12}) & -\mathrm{Im}^T(\mathbf{F}_{22}) & \mathrm{Re}(\mathbf{F}_{22}) \end{bmatrix} \quad (6)$$

is the corresponding Fish information matrix (FIM). In (6),

$$\begin{aligned}\mathbf{F}_{11} = &L(\dot{\mathbf{A}}^H\mathbf{Q}^{-1}\dot{\mathbf{A}}) \odot (\mathbf{B}^*\mathbf{V}^H\mathbf{R}_\Phi^*\mathbf{V}\mathbf{B})+ \\ &L(\dot{\mathbf{A}}^H\mathbf{Q}^{-1}\mathbf{A}) \odot (\mathbf{B}^*\mathbf{V}^H\mathbf{R}_\Phi^*\dot{\mathbf{V}}\mathbf{B})+ \\ &L(\mathbf{A}^H\mathbf{Q}^{-1}\dot{\mathbf{A}}) \odot (\mathbf{B}^*\dot{\mathbf{V}}^H\mathbf{R}_\Phi^*\mathbf{V}\mathbf{B})+ \\ &L(\mathbf{A}^H\mathbf{Q}^{-1}\mathbf{A}) \odot (\mathbf{B}^*\dot{\mathbf{V}}^H\mathbf{R}_\Phi^*\dot{\mathbf{V}}\mathbf{B}),\end{aligned} \quad (7)$$

$$\begin{aligned}\mathbf{F}_{12} =&L(\dot{\mathbf{A}}^H\mathbf{Q}^{-1}\mathbf{A}) \odot (\mathbf{B}^*\mathbf{V}^H\mathbf{R}_\Phi^*\mathbf{V})+ \\ &L(\mathbf{A}^H\mathbf{Q}^{-1}\mathbf{A}) \odot (\mathbf{B}^*\dot{\mathbf{V}}^H\mathbf{R}_\Phi^*\mathbf{V}),\end{aligned} \quad (8)$$

$$\mathbf{F}_{22} = L(\mathbf{A}^H\mathbf{Q}^{-1}\mathbf{A}) \odot (\mathbf{V}^H\mathbf{R}_\Phi^*\mathbf{V}), \quad (9)$$

where $\hat{\mathbf{R}} = \frac{1}{L}\mathbf{X}\mathbf{X}^H$ and $\mathbf{R}_\Phi = \frac{1}{L}\mathbf{\Phi}\mathbf{\Phi}$ are the sample covariance matrices of the received data matrix $\mathbf{X}$ and the transmitted waveforms $\mathbf{\Phi}$, respectively, $(\cdot)^*$ denotes the complex conjugate,

$$\dot{\mathbf{A}} = \left[\frac{\partial \mathbf{a}(\theta_1)}{\partial\theta_1} \quad \cdots \quad \frac{\partial \mathbf{a}(\theta_K)}{\partial\theta_K}\right], \quad (10)$$

and

$$\dot{\mathbf{V}} = \left[\frac{\partial \mathbf{v}(\theta_1)}{\partial\theta_1} \quad \cdots \quad \frac{\partial \mathbf{v}(\theta_K)}{\partial\theta_K}\right]. \quad (11)$$

Note that the CRB matrix depends only on $\mathbf{R}_\Phi$. Therefore, the waveform optimization problem is actually to optimize the sample covariance matrix $\mathbf{R}_\Phi$ of the waveforms.

## IV. WAVEFORM OPTIMIZATION

The MIMO radar waveforms, or more precisely, the sample covariance matrix $\mathbf{R}_\Phi$ of the waveforms, can be optimized, based on the CRB matrix under a total power constraint. The CRB matrix can be calculated approximately, as a function of $\mathbf{R}_\Phi$, using estimated target parameters as well as an estimated covariance matrix of the interference and noise obtained during an initial probing with uncorrelated waveforms. We consider below the waveform optimization for the targets in a particular range bin of interest. Note that the FIM of the target parameters is a linear function of the covariance matrix $\mathbf{R}_\Phi$ of the MIMO radar waveforms. Hence minimizing the trace, the determinant, or the largest eigenvalue of the CRB matrix with respect to $\mathbf{R}_\Phi$ is a convex optimization problem that can be solved efficiently using interior point methods.

First, consider minimizing the trace of the CRB matrix, which is referred to as the *Trace-Opt* criterion:

$$\min_{\mathbf{R}_\Phi} \quad \mathrm{tr}\,(\mathbf{C}) \quad \text{s.t.} \quad \mathbf{R}_\Phi \geq 0, \quad \mathrm{tr}\,(\mathbf{R}_\Phi) = P. \quad (12)$$

Note that in certain practical applications, we may wish to place more emphasis on some target parameters, or perhaps to compensate for the unit selection (such as degrees versus radians for the target angles), or to balance the units used for different target parameters (such as angles versus complex amplitudes). With this in mind, we generalize the Trace-Opt criterion to the following SDP:

$$\begin{aligned}\min_{\{t_k\}_{k=1}^{3K},\mathbf{R}_\Phi} \quad & \sum_{k=1}^{3K} \mu_k t_k \\ \text{s.t.} \quad & \begin{bmatrix} \mathbf{F} & \mathbf{e}_k \\ \mathbf{e}_k^T & t_k \end{bmatrix} \geq 0, \quad k = 1,\cdots,3K, \\ & \mathbf{R}_\Phi \geq 0, \quad \mathrm{tr}\,(\mathbf{R}_\Phi) = P,\end{aligned} \quad (13)$$

where $\{t_k\}$ are auxiliary variables, $\mathbf{e}_k$ denotes the $k$th column of the identity matrix (of dimension $3K$ above), and $\mu_k$, $k = 1,\cdots,3K$, is the $k$th weighting factor. The original Trace-Opt criterion in (12) corresponds to $\mu_k = 1$, $k = 1,\cdots,3K$. Note that the constraints in the above SDP are either linear matrix inequalities (LMIs) or linear equalities in the elements of $\mathbf{R}_\Phi$.

Second, we consider minimizing the determinant of the CRB matrix, which is referred to as the *Det-Opt* criterion. Since the CRB matrix is the inverse of the FIM, minimizing $|\mathbf{C}|$ is equivalent to maximizing $|\mathbf{F}|$:

$$\max_{\mathbf{R}_\Phi} |\mathbf{F}| \quad \text{s.t.} \quad \mathbf{R}_\Phi \geq 0, \quad \mathrm{tr}\,(\mathbf{R}_\Phi) = P. \quad (14)$$

The above max-det problem is a convex optimization problem that can be solved efficiently using public-domain software packages, see [10].

Third, consider minimizing the largest eigenvalue of the CRB matrix, which is referred to as the *Eigen-Opt* criterion. Since minimizing the largest eigenvalue of the CRB matrix $\mathbf{C}$ is equivalent to maximizing the smallest eigenvalue of the FIM $\mathbf{F}$, the waveform optimization under the Eigen-Opt criterion can be readily cast as a SDP:

$$\min_{t,\mathbf{R}_\Phi} -t \quad \text{s.t.} \quad \mathbf{F} \geq t\mathbf{I}, \quad \mathbf{R}_\Phi \geq 0, \quad \mathrm{tr}\,(\mathbf{R}_\Phi) = P, \quad (15)$$

where $t$ is an auxiliary variable and $\mathbf{I}$ denotes the identity matrix (here of dimension $3K$).

In addition to the above design problem, we also consider specifically the problem of minimizing the CRB of one of the target angles only, which we refer to as the *Angle-only* criterion. Note that this criterion was considered in [5] under the spatially and temporally white noise assumption on $\mathbf{Z}$ in (1) and in the single-target case. Without loss of generality, assume that we are interested in the angle of the first target. Then the Angle-only criterion is simply a special case of the generalized Trace-Opt criterion in (13) with $\mu_1 = 1$ and $\mu_k = 0$, $k = 2,\cdots,3K$.

### IV-A. Structure of the Optimal Waveform Covariance Matrix

We show below that for all of the aforementioned optimization criteria and for the total power constraint, the optimized transmitted covariance matrix $\mathbf{R}_\Phi^*$ has a maximum rank of $2K$ (assuming $2K \leq N$). Moreover, its dominant eigenvectors (i.e., the eigenvectors corresponding to its non-zero eigenvalues) belong to the subspace spanned by the columns of $\mathbf{V}$ and $\dot{\mathbf{V}}$.

Let

$$\mathbf{R}_\Phi^* = \boldsymbol{\Delta}\boldsymbol{\Delta}^H. \tag{16}$$

Define

$$\mathbf{U} = [\mathbf{V} \quad \dot{\mathbf{V}}], \tag{17}$$

and decompose $\boldsymbol{\Delta}$ additively as

$$\boldsymbol{\Delta} = \mathbf{P}_U\Delta + \mathbf{P}_U^\perp\Delta, \tag{18}$$

where $\mathbf{P}_U = \mathbf{U}(\mathbf{U}^H\mathbf{U})^{-1}\mathbf{U}^H$, and $\mathbf{P}_U^\perp = \mathbf{I} - \mathbf{P}_U$, with $\mathbf{I}$ denoting the identity matrix. Therefore, we can decompose $\mathbf{R}_\Phi^*$ as a sum of the following two components:

$$\mathbf{R}_\Phi^* \quad = \quad \mathbf{P}_U\Delta\Delta^H\mathbf{P}_U + \tilde{\mathbf{R}}_\Phi. \tag{19}$$

with

$$\tilde{\mathbf{R}}_\Phi = \mathbf{P}_U^\perp\Delta\Delta^H\mathbf{P}_U^\perp + \mathbf{P}_U\Delta\Delta^H\mathbf{P}_U^\perp + \mathbf{P}_U^\perp\Delta\Delta^H\mathbf{P}_U. \tag{20}$$

It can be readily verified that

$$\mathbf{V}^H\tilde{\mathbf{R}}_\Phi\mathbf{V} = \dot{\mathbf{V}}^H\tilde{\mathbf{R}}_\Phi\mathbf{V} = \mathbf{V}^H\tilde{\mathbf{R}}_\Phi\dot{\mathbf{V}} = \dot{\mathbf{V}}^H\tilde{\mathbf{R}}_\Phi\dot{\mathbf{V}} = 0. \tag{21}$$

Substituting (19) into (7), (8), and (9), and observing (21), show that the FIM does not depend on $\tilde{\mathbf{R}}_\Phi$.

Next note that

$$\mathrm{tr}(\tilde{\mathbf{R}}_\Phi) = \mathrm{tr}(\mathbf{P}_U^\perp\Delta\Delta^H\mathbf{P}_U^\perp) = \| \Delta^H\mathbf{P}_U^\perp \|_F^2 \geq 0,$$

where $\| \cdot \|_F$ denotes the Frobenius matrix norm. The equality in (22) holds if and only if $\Delta^H\mathbf{P}_U^\perp = 0$, which implies $\tilde{\mathbf{R}}_\Phi = 0$ (see (20)).

Hence, we have proved that, while the CRB does not depend on $\mathrm{tr}\,\tilde{\mathbf{R}}_\Phi$, an $\tilde{\mathbf{R}}_\Phi \neq 0$ will increase $\mathrm{tr}(\mathbf{R}_\Phi)$ compared with the case of $\tilde{\mathbf{R}}_\Phi = 0$. The conclusion is that under the total power constraint, we necessarily must have $\mathrm{tr}(\tilde{\mathbf{R}}_\Phi) = 0$. Therefore, the optimal $\mathbf{R}_\Phi^*$ can be written as

$$\mathbf{R}_\Phi^* = \mathbf{P}_U\Delta\Delta^H\mathbf{P}_U \triangleq \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^H \tag{22}$$

with $\boldsymbol{\Lambda}$ being a $2K \times 2K$ matrix. By using some matrix properties, the stated result can be concluded readily.

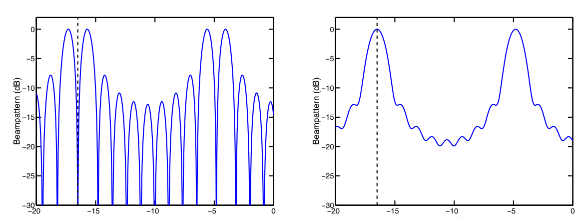
In the numerical example section, we will optimize the CRB based criteria with respect to $\boldsymbol{\Lambda}$ instead of $\mathbf{R}_\Phi^*$ since the dimension of $\boldsymbol{\Lambda}$ is usually much smaller than that of $\mathbf{R}_\Phi^*$, and since the FIM is also a linear function of $\boldsymbol{\Lambda}$. Consequently, the computational complexity can be reduced significantly.

## V. NUMERICAL EXAMPLES

Consider a MIMO radar system with $M = N = 10$ antennas. The distance between adjacent antennas is 0.5-wavelength for the receiving ULA and 5-wavelength for the transmitting ULA. We use an ASNR (Array Signal-to-Noise Ratio) = 40 dB. There is a strong jammer at $-5°$ with an array interference-to-noise ratio (AINR) equal to 100 dB.

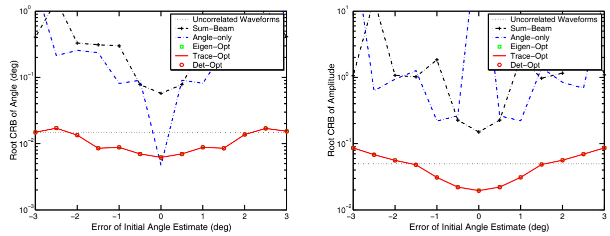
We first consider a single-target case. We assume that the target is at $\theta = -16.5°$ and it has a unit complex amplitude. Figs. 1(a) - 1(b) show the optimized transmit beampatterns obtained using the Angle-only and Trace-Opt in the absence of initial angle estimation error. The beampatterns obtained by Eigen-Opt and Det-Opt are similar to Fig. 1(b). Note that the Angle-only criterion results in a transmit beampattern with a notch at the target angle, while the other criteria place a peak at the target angle.

We consider the effect of initial angle estimation errors on the performance of the waveform optimization. Figs. 2(a) - 2(b) show the root CRB (RCRB) of $\theta$ and $b$ as functions of the error of the initial angle estimate. For comparison purposes, we also show the RCRB when uncorrelated waveforms are transmitted (i.e., $\mathbf{R}_\Phi = (P/N)\mathbf{I}$), and when the sum-beam (i.e., $\mathbf{R}_\Phi = (P/N)\mathbf{v}^*(\theta)\mathbf{v}^T(\theta)$) is used for transmission. Note that the sum-beam yields a higher RCRB than the uncorrelated waveforms. This is because for MIMO Radar, the virtual receive array is a 100-element filled uniform linear array when uncorrelated waveforms are transmitted, which is much larger than the 10-element filled uniform linear array for receiving as a result of sum-beam probing. Note that when the initial angle estimate is accurate, the Angle-only criterion results in the smallest RCRB, which provides around 10 dB improvement compared to the uncorrelated waveforms. However, the approach is very sensitive to the error of the initial angle estimate. Moreover, since this method results in no illumination of the target (see Fig. 1(a)), the corresponding amplitude RCRB is very high. Note also that Trace-Opt, Eigen-Opt and Det-Opt provide a very close performance in this case.




(a) (b)

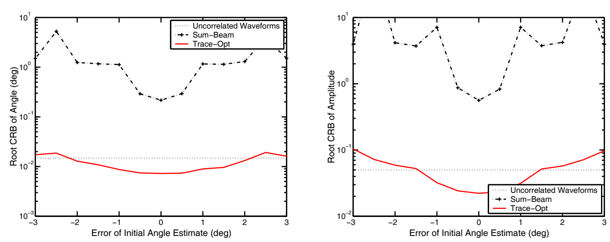
**Fig. 1**. Optimal transmit Beampatterns in the single-target case, formed by (a) Angle-Only and (b) Trace-Opt.




(a) (b)

**Fig. 2**. Root Cramér-Rao bound versus initial angle estimate error for the single-target case. (a) $\theta$ and (b) $b$.

Consider now the two-target case. We assume that $\theta_1 = -16.5°$, $\theta_2 = -10°$, $b_1 = 1$ and $b_2 = 20$. The first target is the target of interest. We found out that the Angle-only criterion is, as before, rather problematic and sensitive to the initial angle estimation error. Fig. 3 shows the RCRB curves for the parameters of the target of interest, i.e., for $\theta_1$ and $b_1$. Note that the optimized waveforms give an approximately 6 dB lower CRB for the parameters of the target of interest than the uncorrelated waveforms. Note also that using the sum-beam gives worse performance than using uncorrelated waveforms. Hence for this case of multiple targets, compared to the previous single-target case, the effect of waveform diversity plays an even more important role than the effect of increased power at the targets provided by the sum-beam transmission.




(a) (b)

**Fig. 3**. Root Cramér-Rao bound versus initial angle estimate errors in $\theta_1$ for the two-target case. (a) $\theta_1$ and (b) $b_1$.

## VI. CONCLUSIONS

We have investigated MIMO radar waveform optimization using several criteria based on the CRB matrix. Numerical examples have been provided to demonstrate the effectiveness of the proposed optimization approaches.

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