ADAPTIVE RADAR WAVEFORM DESIGN FOR MULTIPLE TARGETS: COMPUTATIONAL ASPECTS

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ABSTRACT

In this paper we describe the optimization of an information theoretic criterion for radar waveform design. The method is used to design radar waveforms suitable for simultaneously estimating and tracking parameters of multiple targets. Our approach generalizes the information theoretic water-filling approach of Bell. The paper has two main contributions. First, a new information theoretic design criterion for designing multiple waveforms under a joint power constraint when beamforming is used both at transmitter and receiver. Then we provide a highly efficient algorithm for optimizing the transmitted waveforms, by approximating the information theoretic cost function. We show that using Lagrange relaxation the optimization problem can be decoupled into a parallel set of low-dimensional search problems at each frequency, with dimension defined by the number of targets instead of the number of frequency bands used.

keywords: Adaptive radar, signal design, optimization methods

I. INTRODUCTION

The problem of radar waveform design is of fundamental importance in designing state-of-the-art radar systems. The possibility to vary the transmitted signal on a pulse-by-pulse basis opens the door to great enhancement in estimation and detection capability as well as improved robustness to jamming. Furthermore modern radars can detect and track multiple targets simultaneously. Therefore, designing the transmitted pulses for estimating multiple targets becomes a critical issue in radar waveform design.

Most of existing waveform design literature deals with designs for a single target. One of the important tools in such designs is the use of information theoretic techniques. The pioneering work of Woodward and Davies [1] was the first to suggest that information theoretic tools are important for the development of radar receivers. Following [1] many other works were devoted to the optimization of a single radar waveform for detecting or estimating a single target. See [2] for a detailed review of these results.

Bell [2] was the first to propose using the mutual information between a random extended target and the received signal. His optimization led to a water-filling type strategy. In his paper he assumed that the radar signature is a realization of random Gaussian process with a known power spectral density (PSD). However, when considering real-time signal design we can use his approach to enhance the next transmitted waveform based on the *a priori* known signature. It is interesting to note that Bell's formulation is equivalent to the design of the best communication channel intended to deliver a specific Gaussian random signal (under power constraint on the channel response).

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Whereas the waveform design literature concentrated on the estimation of a single target, modern radars treat multiple targets. Therefore, the development of design techniques for multiple targets is of critical importance to modern radar waveform design.

Recently a great interest has emerged in MIMO radars, where multiple transmit and receive antennas are used with large spatial aperture to overcome target fading (see [3], [4] for many references to the work on MIMO radar including the work of Fuhrmann and the MIT Lincoln lab). However, much less has been done on MIMO waveform design. The only works on waveform design in the MIMO context are by Yang and Blum ([4] and the references therein) and De Maio and Lops [5]. In that work Yang and Blum applied MIMO point-to-point communication theory to design radar waveforms by water-filling the power over the spatial modes of the overall radar scene (channel). They also showed that optimizing the non-causal MMSE and optimizing the mutual information leads to identical results. This finding provides another justification for using the maximum mutual information criterion for the radar waveform design problem. Their work is a novel extension of the work of [2]. However, one should note that by water-filling with respect to the spatial modes, higher power is allocated to the stronger targets. This result is not always desirable, especially in tracking scenarios.

The approach proposed in this paper overcomes this problem, by using the insights provided by multi-user information theory [6] instead of the point-to-point MIMO approach. The various targets are treated as independent signals that need to be estimated, and in the optimization process we provide priorities through a set of priority vectors. A linear combination of the mutual information between each radar beam and its respective target is optimized. This leads to a highly complicated optimization problem strongly related to the interference channel rate region [6]. However, by assuming *linear* pre- and post-processing and an independent estimation of the targets, we are able to reduce the waveform design problem to a problem similar to that of the centralized dynamic spectrum allocation in communication. Furthermore, recent advances in optimization (see [7] and the references therein, [8]) open the way to design techniques specifically tailored for radar waveforms that would be suitable for estimating the parameters of multiple targets.

In this paper we study the problem of radar waveform design for multiple target estimation and tracking based on information theoretic criteria. We treat both the case of a single waveform design and the MIMO case where multiple waveforms should be designed. Our main focus is on adaptive pulse-to-pulse design, where each waveform is designed based on prior information of the target signatures as well as the beamformers design.

The paper has two main contributions: First we extend Bell's results to the design of multiple waveforms for simultaneous estimation and tracking of multiple targets, where the transmitter employs beamforming as well as the receiver. Then, optimization algorithm is proposed. We show that using duality theory the problem can be reduced to a search over a single parameter and parallel low-dimensional optimization problems at each frequency. Interestingly, even though the proposed design criteria for multiple waveforms are non-convex, strong duality [7] still holds asymptotically in the number of frequency bins, which allows us to solve the simpler dual problem. Still we should note that the solution is only approximate since for each fixed number of frequencies the problem is NP-hard [9]

II. INFORMATION THEORETIC APPROACH TO WAVEFORM DESIGN

In this section we extend the waveform design paradigm of Bell [2] to the case of multiple radar transmitters and receivers. The section is divided into three parts: after a brief review of the result of [2] we analyze the case of single waveform design for spatially resolved targets. This is interesting when the transmitter is simple, e.g., in bi-static radar situations. We end up with generalization of our approach to the case of multiple transmit waveforms, each optimized for a specific target. In order to study the trade-off between various radar receivers, we use a linear convex combination of the mutual information between the targets and the received signal at each receiver beam oriented at that specific target.

We begin with a brief overview of Bell's information theoretic approach to the waveform design problem. In this paper we limit ourselves to the case of estimation waveforms for extended targets as described in [2]. We assume that the targets are acting on the transmitted waveform as a random, linear, time-invariant system with discrete-time frequency response taken from a Gaussian ensemble with known PSD. Denote by $\mathbf{h}(f) = [h(f_1), ..., h(f_K)]^T$ the target's radar signature and by $\sigma_h^2(f_k)$ its PSD at frequency f_k. A realization of the received signal is given by

$$x(f_k) = h(f_k)s(f_k) + w(f_k) \quad k = 1, \ldots, K \tag{1}$$

where $s(f_k)$, $w(f_k)$ are respectively the discrete time waveform and clutter at frequency f_k, and K is the number of frequency sub-bands. Under our assumptions and assuming complex envelope signaling over a sufficiently narrow-band division of the transmit bandwidth, the mutual information between the target frequency response and the received signal at frequency f_k is given by

$$I\left(h(f_k); x(f_k)|s(f_k)\right) = \Delta f \log\left(1 + \frac{\sigma_h^2(f_k)|s(f_k)|^2}{\sigma_w^2(f_k)}\right) \tag{2}$$

where $\sigma_w^2(f_k)$ is the clutter PSD at frequency f_k, and Δf is the bandwidth used. The total mutual information between the target frequency response and the received signal is now given by

$$I\left(\mathbf{h}; \mathbf{x}|\mathbf{s}\right) = \Delta f \sum_{k=1}^{K} \log\left(1 + \frac{\sigma_h^2(f_k)|s(f_k)|^2}{\sigma_w^2(f_k)}\right). \tag{3}$$

Bell [2] proved that a water-filling strategy is required to maximize the mutual information, where the transmit PSD is given by

$$|s(f_k)|^2 = \max\left\{0, A - \frac{\sigma_w^2(f_k)}{\sigma_h^2(f_k)}\right\} \tag{4}$$

and A is a constant chosen so that the total power constraint is met. It is interesting to note that unlike the usual communication problem, the waveform design is similar to the optimization of a communication channel for a given signal family rather than the optimization of the signal to achieve capacity.

Bell's approach can now be extended to the design of multiple waveforms suitable for simultaneously estimating multiple targets under a joint total power constraint. Previous work on MIMO radar waveform design [4] put all targets into one large channel matrix, similar to the point-to-point MIMO model. Therefore, it leads to water-filling over the eigen modes of the spatio-temporal channel matrix. This results in emphasis of strong targets. In contrast, we allow for prioritization of targets according to an external design vector α that weighs the various target cost functions. This method generalizes [4], since we are able to allocate more power to targets of interest, even if they are observed only through weak modes of the total channel matrix. Intuitively one can think of our approach as a rate region corresponding to rates of information we observe on various targets. We limit ourselves to linear transmit-receive beamforming, since the common practice in phased-array radars is to perform linear processing. Furthermore, the complete rate region of interference channels is unknown even in the Gaussian noise case. However, since targets are modeled as Gaussian random vectors in this case, we can show that we can approximate the intractable optimization problem by a separable dual optimization problem with a single Lagrange multiplier.

We begin by revising the received signal model. Assume that an array with p elements simultaneously transmits L waveforms. The transmitted signal at frequency f_k is given by

$$\mathbf{t}(f_k) = \sum_{l=1}^{L} \mathbf{u}_\ell(f_k)s_\ell(f_k), \qquad k = 1, ..., K \tag{5}$$

where $\mathbf{u}_\ell(f_k)$ are the beamformer coefficients for the ℓ'th waveform designed for the ℓ'th target at frequency f_k, and $s_\ell(k)$ is the corresponding waveform at frequency f_k. We assume channel reciprocity; i.e., if the receive steering vector is $\mathbf{a}(\theta_\ell, f_k)$, then the transmitted signal arrives at the target with channels $\mathbf{a}^*(\theta_\ell, f_k)$. The signal reflected from the ℓ'th target having signature $\mathbf{h}_\ell = \langle h_\ell(f_k), k = 1, ..., K\rangle$ is therefore given by

$$\mathbf{y}_\ell(f_k) = \sum_{m=1}^{L} \left(\mathbf{a}^*(\theta_\ell, f_k)\mathbf{u}_m(f_k)\right) h_\ell(f_k)s_m(f_k) \tag{6}$$

for $k = 1, ..., K$ (note that we have used index m to enumerate the transmitted waveforms, $m = 1, ..., L$, since ℓ is reserved for the target). Hence, the received signal at the array is given by

$$\mathbf{x}(f_k) = \sum_{m=1}^{L} \mathbf{R}(f_k)\mathbf{u}_m(f_k)s_m(f_k) + \boldsymbol{\nu}(f_k), \tag{7}$$

where $\mathbf{R}(f_k) = \sum_{\ell=1}^{L} \mathbf{R}_\ell(f_k)$. and

$$\mathbf{R}_\ell(f_k) = h_\ell(f_k)\mathbf{a}(\theta_\ell, f_k)\mathbf{a}^*(\theta_\ell, f_k). \tag{8}$$

Assume that a beamformer $\mathbf{w}_\ell(f_k)$ is used to receive the ℓ'th target, resulting in

$$z_\ell(f_k) \quad = \quad \mathbf{w}_\ell^*(f_k) \textstyle\sum_{m=1}^{L} \mathbf{R}(f_k)\mathbf{u}_m(f_k)s_m(f_k) + \nu_\ell'(f_k) \tag{9}$$

where $\nu_\ell'(f_k) = \mathbf{w}_\ell^*(f_k)\boldsymbol{\nu}(f_k)$ is the received noise and clutter component of the ℓ'th beam. Let $\sigma_{\nu_\ell'}^2(f_k) = E\left|\nu_\ell'(f_k)\right|^2 \Delta f$ be the ℓ'th beam noise power at frequency f_k. After algebraic manipulations we can show that the mutual information between the ℓ'th beam and the ℓ'th target at frequency f_k is now given by

$$I_k\left(h_\ell(f_k), z_\ell(f_k)\right) = \log\left(1 + \frac{|z_\ell^t(f_k)|^2}{|z_\ell^n(f_k)|^2 + \sigma_{\nu_\ell'}^2(f_k)}\right)\Delta f. \tag{10}$$

where the signal reflected from the ℓ'th target is denoted by

$$z_\ell^t(f_k) = \sum_{m=1}^{L} \mathbf{w}_\ell^*(f_k)\mathbf{R}_\ell(f_k)\mathbf{u}_m(f_k)s_m(f_k). \tag{11}$$

while the noise and inter-target interference component at the ℓ'th beam is given by

$$z_\ell^n(f_k) = \sum_{n\neq l} \sum_{m=1}^{L} \mathbf{w}_\ell^*(f_k)\mathbf{R}_n(f_k)\mathbf{u}_m(f_k)s_m(f_k) + \nu_\ell'(f_k). \quad (12)$$

We assume that the radar allocates one beam towards each target, since non-linear joint processing of all the beams would lead to an infeasible receiver. Therefore, the total mutual information between the ℓ'th beam and the ℓ'th target is given by:

$$I(\mathbf{h}_\ell; \mathbf{z}_\ell | \mathbf{s}) = \sum_{k=1}^{K} I_k(h_\ell(f_k); z_\ell(f_k)|s(f_k))\,\Delta f \quad (13)$$

where $\mathbf{z}_\ell = \langle z_\ell(f_k) : k = 1, ..., K\rangle$ and $\mathbf{h}_\ell = \langle h_\ell(f_k) : k = 1, ..., K\rangle$ are the received signals using the ℓ'th received beam and the ℓ'th target signature, respectively. $\mathbf{s}_m = [s_m(f_1), ..., s_m(f_K)]^T$ are the signal waveform samples directed towards the m'th target,

$$\mathbf{S} = [\mathbf{s}_1, ..., \mathbf{s}_L]$$

is the complete spatio-temporal waveform matrix, and $\mathbf{s} = \mathrm{vec}(\mathbf{S})$. Assuming that the beamforming vectors are known the multiple waveform design problem is now given by

$$\begin{array}{l} \max_{\mathbf{S}} \sum_{\ell=1}^{L} \alpha_\ell I(\mathbf{h}_\ell; \mathbf{z}_\ell|\mathbf{s}) \\ \text{subject to } \sum_{\ell=1}^{L}\sum_{k=1}^{K} |s_{\ell,k}|^2 \leq P_{\max}, \end{array} \quad (14)$$

where $\boldsymbol{\alpha} = [\alpha_1, ..., \alpha_L]^T$ is the target priority vector. This problem is highly non-linear in the complex waveforms $\mathbf{S}$. Furthermore, it involves cross-correlations between the waveforms, and therefore phase information plays an important role. Hence we need to design not only the waveform spectrum, but the complete complex envelope. The dependence on the phase will have a secondary drawback, since we will not be able to reduce the peak to average of the overall transmitted waveform by properly choosing the waveform phase. However, we will show that in the typical scenario of multiple beams in a large phased array this problem can be approximated by a simpler spectrum design problem.

To conclude the discussion regarding the optimization cost function, we shall comment on the design of the beamformers $\mathbf{w}_\ell^*(f_k), \mathbf{u}_m(f_k)$. There are two approaches to this design. The first employs fixed transmit beams based on classical beamformers. For large arrays typical to phased array radars, this might be sufficient and simple to implement. The receive beams can be easily adapted and will always use an approach similar to MVDR or GSC. The second approach relies on ideas of adaptive transmit beams, exploiting channel state information at the transmitter; i.e., knowledge regarding the locations of the targets can be used to transmit orthogonal beams such that only the illumination of a specific target is received by the adaptive beamformer. This is similar to zero-forcing precoders in MIMO wireless communication. However because of space limitations, these issues will not be discussed further in this paper.

III. WAVEFORM OPTIMIZATION FOR MULTIPLE TARGETS

In this section we discuss the computational aspects of the waveform design problem. The optimization problem in (14) is highly non-linear in all variables and depends also on the correlation between the various waveforms $\mathbf{s}_m$, so we cannot optimize just the power spectral density. This would lead to a completely intractable optimization problem. However, both transmit and receive beamformers are directed towards specific targets, and possibly nulling other targets. Therefore the following approximations hold:

$$\sum_{m=1}^{L} \mathbf{w}_\ell(f_k)\mathbf{R}_\ell(f_k)\mathbf{u}_m'(f_k) \approx \mathbf{w}_\ell(f_k)\mathbf{R}_\ell(f_k)\mathbf{u}_\ell'(f_k) \quad (15)$$

and similarly for $n \neq \ell$

$$\sum_{m=1}^{L} \mathbf{w}_\ell(f_k)\mathbf{R}_n(f_k)\mathbf{u}_m'(f_k) \approx \mathbf{w}_\ell(f_k)\mathbf{R}_m(f_k)\mathbf{u}_m'(f_k) \quad (16)$$

where $\mathbf{u}_m' = \mathbf{u}_m(f_k)s_m(f_k)$. To provide more insight into (15-16), we show that it holds in two typical cases. First, assume that the array is sufficiently large such that $p >> L$. This is typical for systems with hundreds (or thousands) of elements capable of tracking up to several tens of targets. In this case, the energy gain in the main beam of a classical beamformer (with proper windowing) is much higher than the sidelobes. Furthermore, if the radar uses the equivalent of zero-forcing beamforming in the transmit direction, we obtain that each beam is orthogonal to the unintended targets; i.e., $\mathbf{u}_m \perp \mathbf{a}(\theta_n)$ for $m \neq n$, and each waveform is reflected only by its intended target (when $p >> L$, this causes minor degradation). With large arrays, this would cause a minor reduction of the number of degrees of freedom. Similar considerations can hold for the receive beamformer. When applying a MMSE type of beamformer, this would also hold, unless the Gaussian noise were stronger than the interference, in which case we can neglect the contribution of the targets altogether. Therefore, using approximation (15-16), the mutual information (13) now becomes

$$\tilde{I}(\mathbf{h}_\ell; \mathbf{z}_\ell|\mathbf{p}) = \sum_{k=1}^{K} \log\left(1 + \frac{p_{\ell,k}|g_{\ell,\ell}|^2}{\sum_{m\neq\ell} g_{\ell,m}(f_k)p_{m,k} + \sigma^2_{\nu_\ell'}(f_k)}\right)\Delta f, \quad (17)$$

where $\mathbf{p}_m = [p_{m,1}, ..., p_{m,K}]^T$ is the power allocation for the m'th target,

$$\mathbf{P} = [\mathbf{p}_1, ..., \mathbf{p}_L]$$

is the total power allocation matrix, and $\mathbf{p} = \mathrm{vec}(\mathbf{P})$. The constants $g_{\ell,m}$ are defined by

$$\begin{array}{l} g_{\ell,m}(f_k) = \mathbf{w}_\ell^*(f_k)\mathbf{R}_m(f_k)\mathbf{u}_m(f_k) \\ p_{m,k} = |s_m(f_k)|^2\Delta f, \end{array}$$

and include all the prior information regarding the target signatures and the channels.

The problem (14) can now be simplified to

$$\begin{array}{l} \max_{\mathbf{p}} \sum_{\ell=1}^{L} \alpha_\ell \tilde{I}(\mathbf{h}_\ell; \mathbf{z}_\ell|\mathbf{p}) \\ \text{subject to } \sum_{\ell=1}^{L}\sum_{k=1}^{K} p_{\ell,k} \leq P_{\max} \end{array} \quad (18)$$

To solve the multiple waveform design problem, we should note that (18) is a generalized (non-convex) monotropic optimization problem, since the summands of (17) are not concave functions. However, we can show that the time-sharing property [8] holds for (14). This is because adjacent values of k depend continuously on the channel and target coefficients. Assuming that both the beamformer and the target PSD are continuous functions of frequency, the argument of [10] yields the time-sharing property by using frequency sharing of the solutions. Therefore, we have an asymptotically zero duality gap (in the number of frequency bins). Interestingly for any finite K the primal problem is NP-hard [9], and one cannot expect that solving the dual problem will lead to the optimal solution. However, on the practical side, one should note that solving the dual problems, termed Lagrange relaxation, leads to good suboptimal solutions as we will demonstrate in the simulation section.

Applying duality theory we obtain that the Lagrangian dual function is now given by

$$L_d(\lambda) = \inf_{\mathbf{p}} - \sum_{k=1}^{K} L_k(\mathbf{p}_k, \lambda)\,\Delta f + \lambda P_{\max} \quad (19)$$

where

$$L_k(\mathbf{p}_k, \lambda) = \sum_{\ell=1}^{L} \alpha_\ell \tilde{I}_k(\mathbf{h}_\ell(f_k); \mathbf{z}_\ell(f_k)|\mathbf{p}_k)\,\Delta f + \lambda \mathbf{1}^T\mathbf{p}_k. \quad (20)$$

The dual problem now becomes

$$\max_{\lambda \geq 0} \left(\sum_{k=1}^{K} \inf_{\mathbf{p}_k} L_k(\mathbf{p}_k, \lambda) - \lambda P_{\max} \right). \tag{21}$$

Note that unlike the case of a single waveform, we will have multi-dimensional parallel optimization problems. However, this problem has two significant simplifications: The dimension L of each problem is much smaller than the typical number of frequency bins. Second, the problem is unconstrained, which is a major simplification in the non-convex problem.

We can now solve (21) using bi-section search for λ and solving the parallel problems at each frequency given any specific value of λ. This is done using standard unconstrained optimization tools. While the complexity is still large, it is still linear in the number of frequency bins. Furthermore our functions are smooth, and the gradient and Hessian are rational functions. This can be exploited in solving

$$\hat{\mathbf{p}}_k = \inf_{\mathbf{p}_k} L_k(\mathbf{p}_k, \lambda)$$

IV. SIMULATIONS

In this section we present simulated experiments demonstrating the design of multiple waveforms under joint total power constraint using the algorithm of the previous section. We assumed that three targets are present and designed three waveforms transmitted by an omni-directional equispaced linear phased array with 10 elements ($\frac{\lambda}{2}$ spacing) and received by the same array. The target directions were 90^o,160^0 and 20^o, respectively. The number of frequency bins was 100. The receive beamformer used was an MVDR beamformer, and the transmit beamformers were classical beamformers directed towards the targets.Target signatures were Gaussians corresponding to target sizes of 17, 10 and 13 meter respectively. The priority vector was $\boldsymbol{\alpha} = [1, 10, 1]/12$. The transmit-power-to-receive noise ratio was 20 dB, and the targets were centered at 8 GHz. We

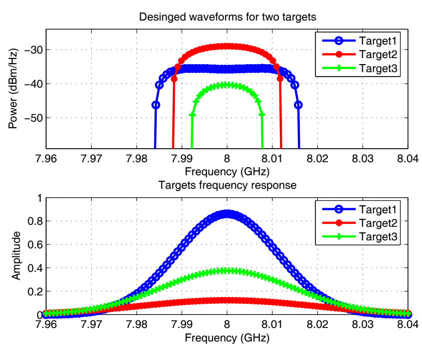

Fig. 1. (a) Designed waveforms (up); (b) Targets' signatures (bottom)

can clearly see that the algorithm designed waveforms centered around 8 GHz with respect to their weights and sizes. In the next experiment, we tested the sensitivity to spatial resolution of the targets. We have used the same target sizes and the same target weights as before. The direction-of-arrival was changed to be 70^o,70.5^o and 71^o respectively, these directions were chosen in order to make a strong interference between targets. In the previous experiment where the targets were spatially resolved we had a large spectral overlap of the designed waveforms. This overlap is caused by the fact that the spatial resolution enables the array to suppress reflections from the other targets therefore allowing better utilization of the frequency domain for both targets. When the targets become close the design criteria reduces the inter-target interference through spectral separation.

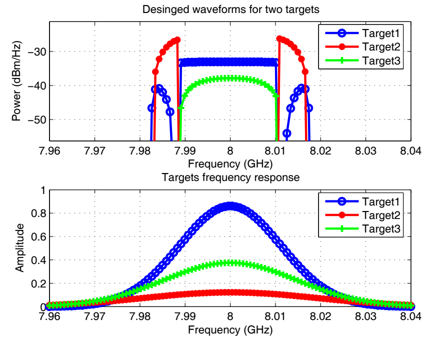

Fig. 2. (a) Designed waveforms. Spatially unresolved targets (solid). Spatially resolved targets (dashed); (b) Targets' signatures (bottom)

V. CONCLUSIONS AND EXTENSIONS

In this paper we have shown that radar waveform design for multiple target estimation can be accomplished using a linear combination of mutual informations between each target signal and the related received beam. Contrary to previous approaches to MIMO radar, we are the first to allow target weighting, by using the analogy of multiuser information theory instead of the point-to-point MIMO model. We then devised a computationally efficient algorithm for solving the problem in the case of a single waveform as well multiple waveforms. We note that similar results can be obtained for the non-causal MMSE design, since in that case the time-sharing property also holds.

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