A GLRT BASED STAP FOR THE RANGE DEPENDENT PROBLEM

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ABSTRACT

We consider in this paper a likelihood principle based approach for the range dependent problem in space time adaptive processing. The proposed generalized likelihood ratio test (GLRT) addresses the range dependent issue by directly applying the likelihood principle to the range dependent signal model. Using the knowledge of platform geometry, we develop maximum likelihood estimators that facilitate the GLRT. This differs from existing methods that rely on data transformations in dealing with the range dependence issue. Numerical examples show that the new GLRT approach exhibits significant performance gain over existing approaches.

***Index Terms*—** Range dependent, General Likelihood Ratio Test, Space Time Adaptive Processing

1. INTRODUCTION

Although shown to achieve good target detection performance, most space-time adaptive processing (STAP) approaches [1, 2] require large amounts of 'target free' training data of secondary range cells for covariance matrix estimation. A perennial issue confronting STAP is the range dependence problem: even with homogeneous clutters, the radar platforms often render the secondary data non IID (independent and identically distributed). Examples include non-side looking monostatic radars, bi- and multi-static radars, and conformal arrays. Exhibited as angle-Doppler dispersion in the frequency domain, the range dependence results in considerable performance degradation for sample matrix inversion (SMI) based STAP even with sufficient secondary data samples.

Significant efforts have been reported to address the range dependence issue [3–5]. Existing approaches include Doppler warping (DW) [3], angle-Doppler compensation (ADC) [4], and registration based methods [5], among others. The above approaches are exclusively based on data transformations, where the range dependent secondary data are transformed, under various criteria, into a data set that are more IID. The transformed data are then used for covariance matrix estimate and subsequently plugged in test statistics that are designed for STAP with IID secondary data. For example, DW applies linear transformation on the secondary data with an attempt to match the Doppler frequencies of secondary data clutter ridge to that of the test cell, with ADC introducing the additional notion of spectral center for Doppler frequency alignment. The registration based method, on the other hand, attempts to realign the entire Doppler-Direction curve to achieve better homogeneity in the transformed data.

While effective compared with conventional STAPs, these data transformation based approaches are rather heuristic. At the very least, there is no guarantee that the transformed secondary data will be truly IID with the target free test cell data. In this paper, we abandon the data transformation regime; instead, we tackle the range dependence problem by directly applying the likelihood principle to the range dependent model itself. In particular, we develop a generalized likelihood ratio test (GLRT) that is derived directly using the original range dependent data set. This new GLRT approach explores the intrinsic structural relationship of the covariance matrices corresponding to different range cells. Not surprisingly, the proposed approach exhibits remarkable performance gain over existing data transformation-based STAP, as demonstrated via numerical simulation.

The paper is organized as follows. In Section II we present the problem formulation for the range dependent STAP, followed by the development of the proposed GLRT. In Section III, we develop the maximum likelihood (ML) estimators that facilitate the implementation of the GLRT. Section IV gives numerical examples that demonstrate significant performance advantages of the proposed GLRT compared with the previous STAP algorithms. We conclude in Section V.

2. THE GLRT FOR THE RANGE DEPENDENT PROBLEM

2.1. Problem Formulation

Consider the following hypothesis testing (HT) problem:

$$\begin{array}{ll}\mathbf{H}_0 & \mathbf{x}=\mathbf{c}\\ \mathbf{H}_1 & \mathbf{x}=\alpha\mathbf{s}+\mathbf{c}\end{array}$$

where $\mathbf{x}$ is the $JN \times 1$ test cell vector with J and N respectively element and pulse numbers; α is the unknown signal amplitude; $\mathbf{s}$ is the known signal (steering) vector; $\mathbf{c}$ is the clutter distributed according to $\mathbf{c} \sim N(0, \mathbf{\Sigma})$. For the current work, we ignore the presence of white Gaussian noise

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and deal only with clutter interference. If $\mathbf{\Sigma}$ is known, the optimum test statistic is $t = |\gamma\left(\mathbf{s}^H\mathbf{\Sigma}^{-1}\right)\mathbf{x}|$, where γ is some constant. However, $\mathbf{\Sigma}$ is typically not known *a priori* and needs to be estimated using target free data from secondary range cells, denoted as $\mathbf{x}_1, \cdots, \mathbf{x}_k$. If the secondary data are IID and have the same distribution as the target cell clutter, one can form the sample covariance matrix: $\hat{\mathbf{\Sigma}} = \frac{1}{k}\sum_{i=1}^{k}\mathbf{x}_i\mathbf{x}_i^H$. For the range dependent problem where $\mathbf{x}_1, \cdots, \mathbf{x}_k$ are no longer IID, sample covariance matrix based STAP test will suffer performance loss.

2.2. Proposed Method

To motivate the proposed GLRT approach, we revisit the classical clutter patch based clutter model [6]. Denote by $\mathbf{\Sigma}_i$ the clutter covariance matrix for the ith range cell and neglect the intrinsic clutter motion, we have

$$\mathbf{\Sigma}_i = \sum_{j=1}^{Nc}\eta_j^i\mathbf{v}_j^i{\mathbf{v}_j^i}^H, \tag{1}$$

where Nc is the total number of discrete clutter patches, $\mathbf{v}_j^i$ and η_j^i are respectively the stearing vector and received clutter power for the jth clutter patch in the ith range cell. Assume that the propagation loss is properly compensated and a homogeneous clutter environment, one can regard η_j^i as constant across all range cell i, denoted by η_j. Clearly, η_j is a function of both the radar cross section of the jth clutter patch and the antenna pattern. With this model, the range dependence is reflected in the distinct steering vectors corresponding to a given clutter patch for different range cells. From [6], $\mathbf{v}_j^i$ is related to spatial and Doppler frequency (${f_s}_j^i$, ${f_d}_j^i$), i.e. $\mathbf{v}_j^i = \mathbf{v}({f_d}_j^i, {f_s}_j^i)$. For a given clutter patch in a particular range cell, its corresponding ${f_d}_j^i$ and ${f_s}_j^i$ are uniquely determined by the platform geometry, i.e., in the case of non-side looking radar, determined by the crab angle (the angle between platform velocity and array orientation) and the cone angle. As such, assuming perfect platform knowledge, these $\mathbf{v}_j^i$'s can be computed and only the power terms η_j are unknown.

With the above model, one can reframe the original HT problem into the following one incorporating the secondary data: for $i = 1, \cdots, k$,

$$\begin{array}{ll}\mathbf{H}_0 & \mathbf{x} \sim CN(0, \mathbf{\Sigma}(\boldsymbol{\eta})), \mathbf{x}_i \sim (0, \mathbf{\Sigma}_i(\boldsymbol{\eta})) \\ \mathbf{H}_1 & \mathbf{x} \sim CN(\alpha\mathbf{s}, \mathbf{\Sigma}(\boldsymbol{\eta})), \mathbf{x}_i \sim (0, \mathbf{\Sigma}_i(\boldsymbol{\eta}))\end{array}$$

where the dependence of the covariance matrices on the unknown $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_{N_c})$ is explicit. With this formulation, the GLRT for the hypothesis testing is

$$LRT = \frac{\max\limits_{\{\alpha,\eta_1,\eta_2,\cdots,\eta_{Nc}\}} f(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_k|\mathbf{H}_1)}{\max\limits_{\{\eta_1,\eta_2,\cdots,\eta_{Nc}\}} f(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_k|\mathbf{H}_0)}. \tag{2}$$

Assume that $\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_k$ are independent, we get,

$$f(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_k|\mathbf{H}_l) = f(\mathbf{x}|\mathbf{H}_l)\prod_{i=1}^{k} f(\mathbf{x}_i|\mathbf{H}_l), \text{ for } l = 0, 1,$$

where

$$\begin{aligned}f(\mathbf{x}_i|\mathbf{H}_l) &= \frac{1}{\pi^{JN}\det(\mathbf{\Sigma}_i)}\exp\left(-\text{tr}\left(\mathbf{\Sigma}_i^{-1}\mathbf{x}_i\mathbf{x}_i^H\right)\right), \quad l = 0, 1 \\ f(\mathbf{x}|\mathbf{H}_0) &= \frac{1}{\pi^{JN}\det(\mathbf{\Sigma})}\exp\left(-\text{tr}\left(\mathbf{\Sigma}^{-1}\mathbf{x}\mathbf{x}^H\right)\right), \\ f(\mathbf{x}|\mathbf{H}_1) &= \frac{1}{\pi^{JN}\det(\mathbf{\Sigma})}\exp\left(-\text{tr}\left(\mathbf{\Sigma}^{-1}(\mathbf{x}-\alpha\mathbf{s})(\mathbf{x}-\alpha\mathbf{s})^H\right)\right)\end{aligned}$$

Next, we develop the ML estimates under both hypotheses to facilitate the implementation of the GLRT.

3. MAXIMUM LIKELIHOOD ESTIMATE

The ML estimation under H_0 and H_1 are classical unconstrained optimization problems. Ideally, one would hope for closed form solutions, which will lead to a closed-form GLRT statistic. For the problems at hand, no closed-form solutions are available. Instead, we will develop a numerical procedure for the estimation problem using the gradient method [7].

We first evaluate the gradient of the log likelihood function over the unknown parameters under both hypotheses. Consider H_0 first. Take partial differentiation with respect to η_j and define $\mathbf{x}_0 \triangleq \mathbf{x}$ for notational reason, we get

$$\begin{aligned}&\frac{\partial}{\partial\eta_j}\log(f(\mathbf{x}, \mathbf{x}_1, \cdots, \mathbf{x}_k|\mathbf{H}_0)) \\ &= \sum_{i=0}^{k}\left(-\frac{\partial}{\partial\eta_j}\log\det\mathbf{\Sigma}_i - \frac{\partial}{\partial\eta_j}\text{tr}\left(\mathbf{\Sigma}_i^{-1}\mathbf{x}_i\mathbf{x}_i^H\right)\right)\end{aligned} \tag{3}$$

In the following, we introduce the following three lemmas in matrix calculus without proof:

Lemma 1 *[8, page 23-24] Let $\mathbf{A}(x)$ be an $n \times n$ matrix, with scalar variable x, then,*

$$\frac{\partial}{\partial x}\triangle(x) = \sum_{k=1}^{n}\triangle_k(x),$$

where $\triangle(x) = \det(\mathbf{A}(x))$;$\triangle_k(x)$ is the determinant formed by replacing the kth row by the row of derivatives,$\{a_{kj}^{'}(x)\}$, $j = 1, \cdots, n$.

Lemma 2 *[8, page 16] Let $\mathbf{A}$ be an $n \times n$ matrix, define the inverse matrix of $\mathbf{A}$ is $\mathbf{A}^{-1}$, then ,*

$$\mathbf{A}^{-1} = [(-1)^{(i+j)}\det(\mathbf{A}_{ij})], i, j = 1, \cdots, n,$$

where $\mathbf{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix formed by deleting row i and column j from $\mathbf{A}$.

$$\frac{\partial}{\partial \eta_j} \log\det\left(\sum_{j=1}^{Nc} \eta_j \mathbf{v}_j^i {\mathbf{v}_j^i}^H\right) = \frac{1}{\det\left(\sum_{j=1}^{Nc} \eta_j \mathbf{v}_j^i {\mathbf{v}_j^i}^H\right)} \sum_{k=1}^{JN} \det\left((\mathbf{v}_j^i {\mathbf{v}_j^i}^H)|_k + \sum_{m\neq k} (\sum_{j=1}^{Nc} \eta_j \mathbf{v}_j^i {\mathbf{v}_j^i}^H)|_m \right) \quad (4)$$

$$\frac{\partial}{\partial \eta_j} \mathrm{tr}\left(\left(\sum_{j=1}^{Nc} \eta_j \mathbf{v}_j^i {\mathbf{v}_j^i}^H\right)^{-1} \mathbf{x}_i \mathbf{x}_i^H\right) = \mathbf{x}_i^H \left(\left[(-1)^{p+q} \det\left(\left(\mathbf{v}_j^i {\mathbf{v}_j^i}^H\right)_{pq}\Big|_k + \sum_{m\neq k} \left(\sum_{j=1}^{Nc} \eta_j \mathbf{v}_j^i {\mathbf{v}_j^i}^H\right)_{pq}\Big|_m \right)\right]\right) \mathbf{x}_i \quad (5)$$

Lemma 3 *Let $\mathbf{A}(x)$ be an $n \times n$ matrix with scalar variable x, and let $\{\mathbf{b}, \mathbf{c}\}$ be $n \times 1$ constant vector with respect to variable x. Then,*

$$\frac{\partial}{\partial \eta_j} \mathbf{b}^H \mathbf{A}(x) \mathbf{c} = \mathbf{b}^H \frac{\partial \mathbf{A}(x)}{\partial \eta_j} \mathbf{c}.$$

Applying the three lemmas, we can obtain Eq. (4,5) shown at the top of this page, where $A|_k$ denotes the $n \times n$ matrix formed by retaining row k and set all the other rows as zero from the $n \times n$ matrix $\mathbf{A}$. Given Eq. (4,5), we can then compute the gradients of the log likelihood functions under H_0 and H_1. The final results are given in Eq. (6-8) at the top of next page, where $\mathbf{\Sigma}_i$ is defined as Eq. 1. With these gradients, the ML estimation can be carried out straightforwardly using the gradient method [7].

4. NUMERICAL ANALYSIS

In this section, we evaluate the detection performance of the proposed GLRT via numerical examples and compare it with that of the CFAR detector in [2]. The CFAR test in [2] was developed by first assuming covariance matrix known and then replacing it with estimated covariance matrix. Here we consider three methods for covariance matrix estimation:

- Sample covariance matrix using the original data. This is the original CFAR test in [2].
- Sample covariance matrix using the data transformed via Angle Doppler compensation [4].
- The ML estimate developed using the proposed approach in Section 3 by discarding the test cell data.

Of particular interest is the last one, where we plug our ML estimate into a standard STAP test, which is termed as a Mixed Approach. We will see that while this outperforms other covariance matrix estimate, the performance still comes far below the true GLRT approach.

We assume a four antenna two pulse (hence $JN = 8$ dimensions) configuration with a crab angle $\pi/6$. Assume that the total number of clutter patches is $N_c = 16$ which is **known** to the estimator. The secondary data size is 30.

Fig. 1 is the receiver operating characteristic (ROC) curve of the four test approaches. While ADC does show reasonable improvement over SMI, the mixed approach outperforms both SMI and ADC; its advantage comes with a more accurate, *albeit* complex, estimation of the covariance matrix. More importantly, the proposed GLRT outperforms all three alternatives by a significant margin, demonstrating the advantage of applying the likelihood principle directly to the data model.

The clutter patch based covariance model (1) is only modelling tool for the actual clutter covariance. For this model to be accurate, N_c needs to be sufficiently large and is typically unknown to the STAP processor. For this purpose, we test the detection performance where the data is generated using $N_c = 100$ clutter patches, whereas the ML estimate only assumes $N_c = 16$, i.e., one will estimate the 16 'virtual' clutter patches that are representative of the true clutter covariance matrix. The results are given in Fig. 2. The proposed GLRT has very robust performance gain over the other three methods, where a standard STAP was used with various covariance matrix estimate.

5. CONCLUSION

In this paper, we describe some initial attempts to develop a new detection paradigm for the range dependent problem that exploits the structural knowledge of clutter covariance matrix. The proposed GLRT differs from existing approaches in that it abandons the heuristic data transformation framework in favor of a systematic likelihood principle based approach. The potential gain in detection performance over previous approaches for the range dependence problem is enormous, as demonstrated by the numerical simulations. Indeed, initial results using P_d over SNR plot (not included due to space limit) with low dimensional cases indicated an SNR gain of $> 5dB$ over existing approaches.

While the performance gain is promising, the complexity of the ML estimation presents a significant challenge for high dimensional case. Our future research will be focused on finding sub-optimal procedures to make the computation more affordable. An interesting direction is to integrate the likelihood principle based framework with joint domain localized (JDL) detector, which is both data and computationally efficient. We will also extend the developed framework to more complicated cases, in terms of both clutter model (e.g., incorporating receiver noise) and radar platforms (e.g., bi-/multi-static radars and conformal arrays).

$$\frac{\partial}{\partial \eta_j}\log(f(\mathbf{x},\mathbf{x}_1,\cdots,\mathbf{x}_k|\mathbf{H}_0)) = -\sum_{i=0}^{k}\left(\frac{1}{\det(\mathbf{\Sigma}_i)}\sum_{k=1}^{JN}\det\left((\mathbf{v}_j^i\mathbf{v}_j^{i\,H})|_k + \sum_{m\neq k}(\mathbf{\Sigma}_i)|_m\right)\right.$$
$$\left. +\mathbf{x}_i^H\left(\left[(-1)^{p+q}\det\left(\left(\mathbf{v}_j^i\mathbf{v}_j^{i\,H}\right)_{pq}|_k + \sum_{m\neq k}\left((\mathbf{\Sigma}_i)_{pq}|_m\right)\right)\right]\right)\mathbf{x}_i\right) \quad (6)$$

$$\frac{\partial}{\partial \eta_j}\log(f(\mathbf{x},\mathbf{x}_1,\cdots,\mathbf{x}_k|\mathbf{H}_1)) = -\frac{1}{\det(\mathbf{\Sigma})}\sum_{k=1}^{JN}\det\left((\mathbf{v}_j\mathbf{v}_j{}^H)|_k + \sum_{m\neq k}(\mathbf{\Sigma})|_m\right)$$
$$-(\mathbf{x}-\alpha\mathbf{s})^H\left(\left[(-1)^{p+q}\det\left(\left(\mathbf{v}_j\mathbf{v}_j{}^H\right)_{pq}\Big|_k + \sum_{m\neq k}(\mathbf{\Sigma}_i)_{pq}|_m\right)\right]\right)(\mathbf{x}-\alpha\mathbf{s})$$
$$-\sum_{i=1}^{k}\left(\frac{1}{\det(\mathbf{\Sigma}_i)}\sum_{k=1}^{JN}\det\left((\mathbf{v}_j^i\mathbf{v}_j^{i\,H})|_k + \sum_{m\neq k}(\mathbf{\Sigma}_i)|_m\right)\right.$$
$$\left. +\mathbf{x}_i^H\left(\left[(-1)^{p+q}\det\left(\left(\mathbf{v}_j^i\mathbf{v}_j^{i\,H}\right)_{pq}|_k + \sum_{m\neq k}\left((\mathbf{\Sigma}_i)_{pq}|_m\right)\right)\right]\right)\mathbf{x}_i\right) \quad (7)$$

$$\frac{\partial}{\partial \alpha}\log(f(\mathbf{x},\mathbf{x}_1,\cdots,\mathbf{x}_k|\mathbf{H}_1)) = \mathbf{s}^H(\mathbf{\Sigma})^{-1}\mathbf{x} + \mathbf{x}^H(\mathbf{\Sigma})^{-1}\mathbf{s} - 2\alpha\mathbf{s}^H(\mathbf{\Sigma})^{-1}\mathbf{s} \quad (8)$$

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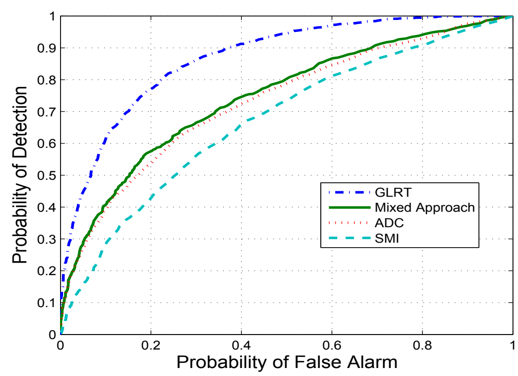

Fig. 1. The (ROC) curve for $N_c = 16$.

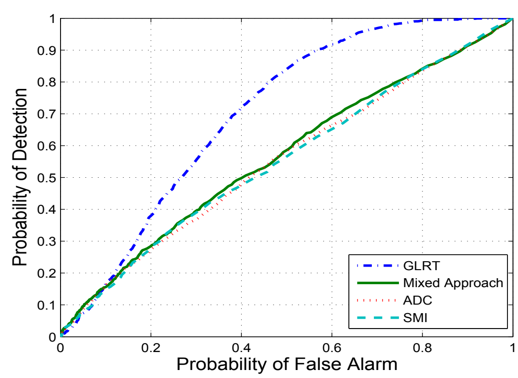

Fig. 2. The ROC curve for $N_c = 100$ while the MLE uses $N_c = 16$.