CROSS COHERENCE AND JOINT PDF OF THE BARTLETT AND CAPON POWER SPECTRAL ESTIMATES

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ABSTRACT

The Bartlett algorithm results from a conventional (Fourier or beamforming) approach to power spectral estimation and the Capon algorithm results from an adaptive approach. Both algorithms make use of the data sample covariance matrix (SCM). The Bartlett algorithm relies directly on the SCM, while the Capon approach relies on the inverse of the SCM. Since both statistics depend on the same data, they are not independent in general. While the marginal distribution of each statistic is well-known, the joint dependence is unknown. This paper presents a complete statistical summary of the joint dependence of the Bartlett and Capon statistics, showing that the dependence is expressible via a 2×2 complex Wishart matrix where the coupling is determined by a single measure of coherence defined herein. Interestingly, this measure of coherence leads to a new two-dimensional algorithm capable of yielding better resolution than the Capon algorithm.

Index Terms–Adaptive, Bartlett, beamforming, Capon, coherence, conventional, cross-spectra, joint pdf, resolution, two-dimensional.

1. INTRODUCTION

The Bartlett algorithm results from a conventional (Fourier or beamforming) approach to power spectral estimation and the Capon algorithm results from an adaptive approach. Both algorithms make use of the data sample covariance matrix (SCM). The Bartlett algorithm relies directly on the SCM, while the Capon approach relies on the inverse of the SCM. Since both statistics depend on the same data, they are not independent in general. While the marginal distribution of each statistic is well-known [1], the joint dependence is unknown. Techniques such as diagonal loading help in the low sample support case [5], but likewise engage the trade-space between conventional and adaptive approaches. Can this trade-space be characterized statistically? This is a difficult task in general, but this paper provides a start. Herein a complete statistical summary of the joint dependence of the Bartlett and Capon statistics is presented, showing that the dependence is expressible via a 2×2 complex Wishart matrix where the coupling is determined by a single measure of coherence defined herein. Interestingly, this measure of coherence naturally suggests a new two-dimensional algorithm yielding better resolution than the Capon algorithm.

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2. POWER SPECTRAL ESTIMATION

The Capon and Bartlett algorithms represent filterbank approaches to spectral estimation [10]. Data is obtained from an array of N sensors distributed in space. Each data observation taken across the array (called a spatial snapshot) is modeled as an $N \times 1$ zero mean complex Gaussian vector[1] with representation $\mathbf{x} = S\mathbf{v}(\theta) + \mathbf{n}$ where the array response of the signal of interest associated with direction-of-arrival (DOA) parameter θ is given by $\mathbf{v}(\theta)$, and its complex amplitude is Gaussian distributed such that $S \sim \mathcal{CN}_1(0, \sigma_S^2)$, and the colored noise is denoted by $\mathbf{n}$ having covariance $E\{\mathbf{nn}^H\} = \mathbf{R}_N$, where $E\{\cdot\}$ denotes the statistical expectation. Note that $E\{\mathbf{x}\} = \mathbf{0}$ and that $E\{\mathbf{xx}^H\} = \mathbf{R} = \mathbf{R}_N + \sigma_S^2\mathbf{v}(\theta)\mathbf{v}^H(\theta)$. A finite set of L array observations is accrued over time and assembled in a data matrix: $\mathbf{X} = [\mathbf{x}(1)|\mathbf{x}(2)|\cdots|\mathbf{x}(L)]$, where $\mathbf{x}(l) \sim \mathcal{CN}_N(\mathbf{0}, \mathbf{R})$, $l = 1, 2, \ldots, L$. The spatial snapshots are used to form the unnormalized data spatial covariance estimate $\widehat{\mathbf{R}} = \mathbf{XX}^H$, the unnormalized SCM, from which the Capon and Bartlett algorithms generate spatial power spectral estimates. It shall be assumed herein that $L \geq N$ to guarantee SCM invertibility.

3. THE BARTLETT AND CAPON ALGORITHMS

The Capon and Bartlett DOA angle estimates are obtained as the arguments of the largest peaks of the estimated spatial power spectra. If $P(\theta)$ represents the estimated spectrum as a function of angle, then the maximum output provides an estimate of the signal power σ_S^2, and the signal DOA estimate is given by the scan value of θ that achieves this maximum; namely,

$$\widehat{\theta} = \arg\max_{\theta} P(\theta) \tag{1}$$

(assuming a single signal is present). It shall be assumed that K signals are present in the data, and that the Capon/Bartlett parameter estimates $\widehat{\theta}_k$, $k = 1, 2, \ldots, K$ are obtained as the arguments of the K largest peaks of $P(\theta)$.

[1] The notational convention adopted is as follows: italics indicates a scalar quantity, as in A; lower case boldface indicates a vector quantity, as in $\mathbf{a}$; upper case boldface indicates a matrix quantity, as in $\mathbf{A}$. The n-th row and m-th column of matrix $\mathbf{A}$ will be indicated by $[\mathbf{A}]_{n,m}$. Variables will be assumed complex in general, but some will be real (obvious from context). Re(A) is the real part of A and Im(A) is the imaginary part. The complex conjugation of a quantity is indicated by a superscript $*$ as in A^*. The matrix transpose is indicated by a superscript T as in $\mathbf{A}^T$, and the complex conjugate plus matrix transpose is indicated by a superscript H as in $\mathbf{A}^H = (\mathbf{A}^T)^*$.

3.1. The Bartlett Algorithm

Let the conventional beamforming weight steered to angle θ for an array with element spatial locations $\mathbf{z}_n$, $n = 1, 2, \ldots, N$ be given by $\mathbf{v}(\theta) = [e^{j\mathbf{k}_\theta^T \mathbf{z}_1}, e^{j\mathbf{k}_\theta^T \mathbf{z}_2}, \ldots, e^{j\mathbf{k}_\theta^T \mathbf{z}_N}]^T$, where $\mathbf{k}_\theta = (2\pi/\lambda)\mathbf{a}(\theta)$ is the wavenumber vector, $\mathbf{a}(\theta)$ is the 3×1 unit vector pointing in the assumed direction of field propagation, $\lambda = c/f$ is the wavelength at temporal frequency f, and c is the wave propagation speed. Note that for a uniform linear array (ULA), this spatial filter weight can be easily implemented with the Fast Fourier Transform (FFT). The Bartlett spectral estimate evaluated at spatial frequency (or angle) θ is given by

$$P_{Bartlett}(\theta) = \frac{1}{L}\sum_{l=1}^{L}\left|\mathbf{v}^H(\theta)\mathbf{x}(l)\right|^2 = \frac{1}{L}\cdot\mathbf{v}^H(\theta)\widehat{\mathbf{R}}\mathbf{v}(\theta) \quad (2)$$

with ambiguity function defined as $\psi_{Bartlett}(\theta) \triangleq \mathbf{v}^H(\theta)\mathbf{R}\mathbf{v}(\theta)$.

3.2. The Capon Algorithm

Capon proposed the following constrained optimization problem for the filterbank weight vector $\mathbf{w}$

$$\min_{\mathbf{w}} \mathbf{w}^H\mathbf{R}\mathbf{w} \text{ such that } \mathbf{w}^H\mathbf{v}(\theta) = 1 \quad (3)$$
$$\Longrightarrow \mathbf{w}_{MVDR} = \frac{\mathbf{R}^{-1}\mathbf{v}(\theta)}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)},$$

that yields the well known minimum variance distortionless response (MVDR) filter as its solution. Note that the optimal filter weight depends on the data covariance matrix $\mathbf{R}$. By construction, this optimal filter will cancel all spatially coherent energy from all directions other than the scan direction θ. The average output power is given by

$$\begin{aligned} E\left\{\left|\mathbf{w}_{MVDR}^H\mathbf{x}\right|^2\right\} &= \frac{1}{\mathbf{v}^H(\theta)\mathbf{R}^{-1}\mathbf{v}(\theta)} \triangleq \psi_{Capon}(\theta) \\ &= \frac{1}{\mathbf{v}^H(\theta)\mathbf{R}_N^{-1}\mathbf{v}(\theta)} + \sigma_S^2 \end{aligned} \quad (4)$$

where the Capon ambiguity function $\psi_{Capon}(\theta)$ has been defined. The last equality in (4) holds only when the signal array response in $\mathbf{R}$ perfectly matches that used to form the weight vector, and when $\mathbf{R}$ is perfectly known. Capon, therefore, reasoned that for large enough sample support, an estimate of the covariance $\mathbf{R}$ can be used with (4) to estimate the signal power σ_S^2 and the corresponding signal parameter θ. Using the covariance estimate $\widehat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H$, Capon proposed the following power spectral estimator

$$P_{Capon}(\theta) = \frac{1}{L-N+1}\cdot\frac{1}{\mathbf{v}^H(\theta)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta)}. \quad (5)$$

$P_{Capon}(\theta)$ can be further normalized to ensure that it is a true power spectral density.

3.3. Properties of Estimators

The reader is referred to [10] for a detailed discussion of properties and contrasts between conventional Bartlett and adaptive Capon approach. It is noted briefly herein that the Bartlett estimator has resolution capability limited by the aperture length (Fourier limit) independent of signal-to-noise ratio (SNR), and due to its vulnerability in the sidelobes various windows/tapers have been proposed for sidelobe control. The Capon estimator, however, has superior resolution ability that improves with SNR, and it adaptively nulls any energy in the sidelobes while passing the direction of interest undistorted.

Both the Bartlett and Capon statistics have marginal distributions that are complex chi-squared [1, 9].

4. CROSS COHERENCE AND JOINT PDF OF SPECTRAL ESTIMATES

Define the filter weight steered to angle θ_b for the Bartlett algorithm as $\mathbf{w}_B = \mathbf{v}(\theta_b)$ and its output from a single snapshot as $y_B(\theta_b)$. Define the clairvoyant filter weight steered to θ_a for the Capon algorithm as $\mathbf{w}_C = \mathbf{R}^{-1}\mathbf{v}(\theta_a)/\mathbf{v}^H(\theta_a)\mathbf{R}^{-1}\mathbf{v}(\theta_a)$ and its output based on the same spatial snapshot as $y_C(\theta_a)$:

$$\begin{aligned} \mathbf{x} \longrightarrow \boxed{\mathbf{w}_C} \longrightarrow \mathbf{w}_C^H\mathbf{x} = y_C(\theta_a) \\ \mathbf{x} \longrightarrow \boxed{\mathbf{w}_B} \longrightarrow \mathbf{w}_B^H\mathbf{x} = y_B(\theta_b). \end{aligned} \quad (6)$$

The cross coherence between these two filters is given by

$$\boxed{\begin{aligned} \cos^2\phi &\triangleq \frac{E\left\{|y_B(\theta_b)\cdot y_C^*(\theta_a)|^2\right\}}{E\left\{|y_B(\theta_b)|^2\right\}\cdot E\left\{|y_C(\theta_a)|^2\right\}} \\ &= \frac{\psi_{Capon}(\theta_a)}{\psi_{Bartlett}(\theta_b)}\left|\mathbf{v}^H(\theta_a)\mathbf{v}(\theta_b)\right|^2. \end{aligned}} \quad (7)$$

As implied by the denotation this measure of cross coherence can likewise be interpreted as a generalized cosine [2] between the optimal adaptive beamformer weight vector, *i.e.* any $\propto \mathbf{R}^{-1}\mathbf{v}(\theta_a)$, and a non-adaptive conventional beamforming weight vector, *i.e.* any $\propto \mathbf{v}(\theta_b)$, where the metric space is defined by the true data covariance $\mathbf{R}$, allowing one to also define the generalized sine $\sin^2\phi \triangleq 1 - \cos^2\phi$. Because of the weighting by the data covariance, this cosine can be less than unity even when the array responses are steered to the same direction, *i.e.* when $\theta_a = \theta_b$.

The statistical dependence between the SCM based Bartlett and Capon spectral estimates is ultimately determined by this measure of cross coherence [7]. A summary is provided in the following subsections using the related but equivalent statistics below:

$$\begin{aligned} \tilde{P}_{Capon}(\theta_a) &\triangleq (L-N+1)\cdot P_{Capon}(\theta_a), \\ \tilde{P}_{Bartlett}(\theta_b) &\triangleq L\cdot P_{Bartlett}(\theta_b). \end{aligned} \quad (8)$$

4.1. Partially Coherent Spectral Estimates: $0 < \cos^2\phi < 1$

Define the following 2×2 matrix

$$\mathbf{\Xi} \triangleq \begin{bmatrix} \sqrt{\psi_{Capon}(\theta_a)} & \mathbf{v}^H(\theta_a)\mathbf{v}(\theta_b)\sqrt{\psi_{Capon}(\theta_a)} \\ 0 & \sqrt{\psi_{Bartlett}(\theta_b)\cdot\sin^2\phi} \end{bmatrix}. \quad (9)$$

Let $\mathbf{\Theta} \sim \mathcal{CW}(L-N+1, \mathbf{I}_2)$, i.e. standardized central 2×2 complex Wishart distributed matrix. If $\mathbf{\Delta} \triangleq \mathbf{\Xi}^H\mathbf{\Theta}\mathbf{\Xi}$, then it follows that [4, 9]

$$\mathbf{\Delta} \sim \mathcal{CW}\left(L-N+1, \mathbf{\Xi}^H\mathbf{\Xi}\right) \text{ where} \quad (10)$$

$$\boldsymbol{\Xi}^H\boldsymbol{\Xi} = \psi_{Capon}(\theta_a)\begin{bmatrix} 1 & \mathbf{v}^H(\theta_a)\mathbf{v}(\theta_b) \\ \mathbf{v}^H(\theta_b)\mathbf{v}(\theta_a) & \dfrac{\psi_{Bartlett}(\theta_b)}{\psi_{Capon}(\theta_a)} \end{bmatrix}. \quad (11)$$

It can be shown [7] that an equivalent stochastic representation for the Capon and Bartlett estimates when based on the same SCM is given by[2]

$$\begin{aligned} \tilde{P}_{Capon}(\theta_a) &\overset{d}{=} [\boldsymbol{\Delta}]_{1,1} \\ \tilde{P}_{Bartlett}(\theta_b) &\overset{d}{=} [\boldsymbol{\Delta}]_{2,2} + \chi^2_{N-1}\cdot\psi_{Bartlett}(\theta_b) \end{aligned} \quad (12)$$

where χ^2_{N-1} is a complex chi-squared statistic of $N-1$ degrees of freedom [9] independent of $\boldsymbol{\Delta}$. This statistical summary shows that the adaptive and conventional spectral estimates are coupled via a 2×2 complex Wishart matrix. It also represents a generalization of the Capon-Goodman result in [1], since it also conveys joint dependence and not just the marginal distributions of each statistic.

As an example of the utility of (12), note that the correlation coefficient of the SCM based spectral estimates, *i.e.*

$$\rho_{CB} \triangleq \frac{\text{cov}\left[P_{Capon}(\theta_a), P_{Bartlett}(\theta_b)\right]}{\sqrt{\text{var}\left[P_{Capon}(\theta_a)\right]\cdot\text{var}\left[P_{Bartlett}(\theta_b)\right]}} \quad (13)$$

is obtainable using known moments of complex Wishart matrices [3] and given exactly by

$$\rho_{CB} = \sqrt{\frac{L-N+1}{L}}\cdot\cos^2\phi. \quad (14)$$

4.2. Perfectly Coherent Spectral Estimates: $\cos^2\phi = 1$

By the Schwartz inequality it follows that $\cos^2\phi = 1$, and therefore $\sin^2\phi = 0$, if and only if $\mathbf{Rv}(\theta_b) \propto \mathbf{v}(\theta_a)$. Note that $\boldsymbol{\Xi}$ will not be full rank in this case. It can be shown [7] that in this case the stochastic representation in (12) simplifies to

$$\begin{aligned} \tilde{P}_{Capon}(\theta_a) &\overset{d}{=} {}_c\chi^2_{L-N+1}\cdot\psi_{Capon}(\theta_a) \\ \tilde{P}_{Bartlett}(\theta_b) &\overset{d}{=} \left[{}_c\chi^2_{L-N+1} + \chi^2_{N-1}\right] \times \\ &\quad\ \tau\cdot\psi_{Bartlett}(\theta_b) \end{aligned} \quad (15)$$

where ${}_c\chi^2_{L-N+1}$ and χ^2_L are independent complex chi-squared statistics. The left subscript c on ${}_c\chi^2_{L-N+1}$ is a tag to indicate that the same random variable is being referenced, and τ is real scalar related to the constant of proportionality for the general case of $\mathbf{Rv}(\theta_b) \propto \mathbf{v}(\theta_a)$.

4.3. Perfectly Incoherent Spectral Estimates: $\cos^2\phi = 0$

Note that if $\mathbf{v}^H(\theta_a)\mathbf{v}(\theta_b) = 0$, *i.e.* the steering vectors are mutually orthogonal, then $\cos^2\phi = 0$ from which it follows [7] that $\boldsymbol{\Xi}^H\boldsymbol{\Xi}$ is diagonal and the Capon and Bartlett spectral estimates are not only uncorrelated, but are also statistically independent even though based on the same SCM. The statistical summary becomes

$$\begin{aligned} \tilde{P}_{Capon}(\theta_a) &\overset{d}{=} \chi^2_{L-N+1}\cdot\psi_{Capon}(\theta_a) \\ \tilde{P}_{Bartlett}(\theta_b) &\overset{d}{=} \chi^2_L\cdot\psi_{Bartlett}(\theta_b) \end{aligned} \quad (16)$$

where χ^2_{L-N+1} and χ^2_L are independent complex chi-squared statistics. Clearly, the correlation in the sidelobe structure of the beampattern is the major determinant of the level of coherence between the Capon and Bartlett estimates.

4.4. A New 2-D Algorithm: A Cross Spectral Estimate

Since the cross coherence defined in (7) clearly plays a critical role in the describing the dependence between these two spectral estimates, it is reasonable to consider what can be learned or inferred from a SCM based estimate of this measure. Define the SCM based estimate of the cross-spectra of the Capon and Bartlett algorithms as

$$P_{CB}(\theta_a,\theta_b) = \frac{\left|\mathbf{v}^H(\theta_a)\mathbf{v}(\theta_b)\right|^2\left[L/(L-N+1)\right]}{\mathbf{v}^H(\theta_a)\widehat{\mathbf{R}}^{-1}\mathbf{v}(\theta_a)\cdot\mathbf{v}^H(\theta_b)\widehat{\mathbf{R}}\mathbf{v}(\theta_b)}. \quad (17)$$

The distribution of this new statistics can be determined from (12) (see [7]). In the next section, its use in resolving signals is demonstrated.

5. NUMERICAL EXAMPLES

Consider a Direction of Arrival (DOA) estimation scenario involving two equal power planewave sources and a set of signal bearing snapshots

$$\mathbf{x}(l) \sim \mathcal{CN}\left[\mathbf{0}, \mathbf{I} + \sum_{k=1}^{2}\sigma^2_{S_k}\mathbf{v}(\theta_k)\mathbf{v}^H(\theta_k)\right] \quad (18)$$

$l = 1,2,\ldots,L$ for an $N = 18$ element uniform linear array (ULA) with slightly less than $\lambda/2$ element spacing, and $L = 3N$ snapshots. The array has a 3dB beamwidth of 7.2 degrees and the desired target signals are placed at $\theta_1 = 90$ degrees (array broadside), $\theta_2 = 93$ degrees (less than half a beamwidth separation). Figure 1 illustrates the cross spectrum estimate $P_{CB}(\theta_a,\theta_b)$ shown by the gray color scale. Plotted on top of this image are normalized versions of the Capon and MUSIC algorithm 1-D spectra for comparison. This is an example for which the MUSIC algorithm was able to resolve two signals at $\sigma^2_{S_k} = 1$ dB, but the Capon algorithm has failed. The cross spectrum, however, shows two distinct peaks located at the same angles as the two peaks of the MUSIC algorithm, but obtained without knowledge of the rank of the signal subspace. This is clearly a single realization in which the true probability of resolution for the Capon algorithm is a little less than 10 percent. Averaging over several realizations, an emprically based estimate of the probability of resolution can be obtained for all three algorithms. Using same parameters, the results are illustrated in Figure 2 and based on 150 Monte Carlo simulations for each SNR point. Included for comparison is the theoretical two-point probability of resolution derived for the Capon algorithm (black circles) [6]. The two signals were declared resolved for the Capon and MUSIC algorithm if two peakes appeared in the spectrum within the range of 88 to 95 degrees. For the cross-spectrum the two signals were declared resolved if two peaks appeared in

[2] If random variable A has the same probability density function as random variable B, then they are said to be equal in distribution and this is denoted by $A \overset{d}{=} B$.

the slanted slither of a ± 7 degree band of Bartlett angles about the diagonl of the spectrum, but within the Capon angle range of 88 to 95 degrees. This figure shows significant improvement of resolution by the cross spectrum over both the Capon algorithm and MUSIC for this particular example.

Detailed theoretical analysis of this algorithm is ongoing [7], however, these initial results are encouraging. Additional analysis and further examples will be presented at the conference.

6. CONCLUSIONS

This paper provides a complete statistical summary of the joint dependence of the Bartlett and Capon power spectral statistics, showing that the coupling is expressible via a 2×2 complex Wishart matrix where the degree coupling is determined by a single measure of cross coherence defined herein. This measure of coherence leads to a new two-dimensional algorithm capable of yielding signficantly better resolution than the Capon algorithm, often commensurate wit but at times exceeding finite sample based MUSIC. Initial numerical results are encouraging and theoretical analysis is ongoing.

7. REFERENCES

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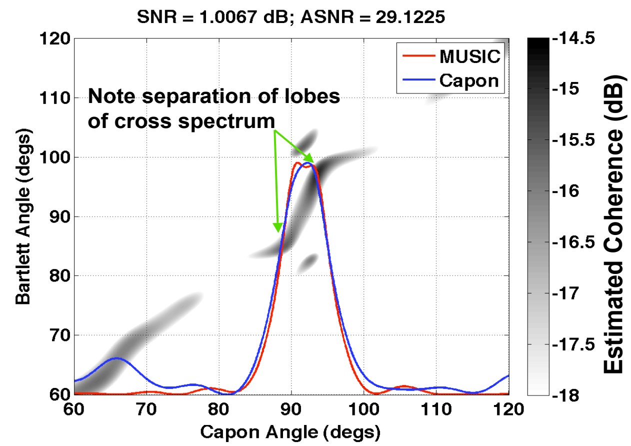

Fig. 1. Single realization Demonstrating Resolution of Cross-Spectrum

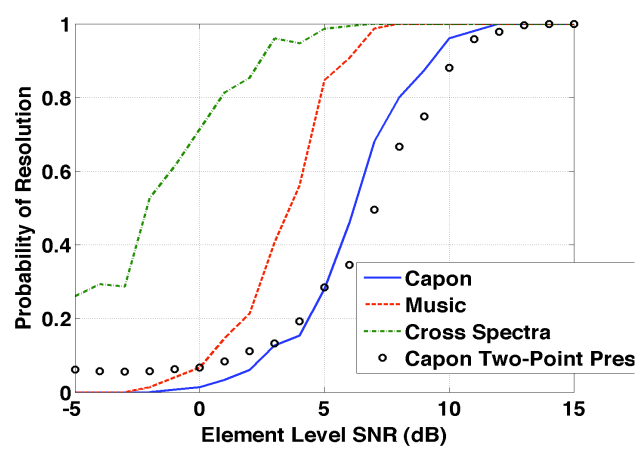

Fig. 2. Probability of Resolution of Cross-Spectrum vs. SNR