OPTIMAL ARRAY PATTERN SYNTHESIS VIA MATRIX WEIGHTING

Yao Xie, Jian Li, Xiayu Zheng

University of Florida
Dept. of Electrical and Computer Engineering
Gainesville, FL 32611, USA.

James Ward

MIT Lincoln Laboratory
244 Wood Street
Lexington, MA 02420, USA

ABSTRACT

We present new array beampattern synthesis approaches via semidefinite relaxation (SDR) for arbitrary array. Compared to the conventional approaches of using weight vectors at the array output for array pattern synthesis, which we refer to as the Vector Weighting Approaches (VWA), weight matrices are used at the array output by MWA for much improved flexibility for optimal array pattern synthesis, and globally optimal solutions can be determined efficiently due to convex optimization formulations. Numerical examples are presented to show the excellent performance of MWA.

Index Terms – Array Signal Processing

1. INTRODUCTION

One of the fundamental problems in array signal processing is array pattern synthesis. The conventional formulation for this problem, which we referred to as the Vector Weighting Approaches (VWA), as shown in Fig. 1(a), is to design a vector of complex-valued weights for the sensor outputs and coherently sum up the weighted signals to form a desired beampattern [1–3].

Recently, numerical approaches based on convex optimization techniques for array pattern synthesis using VWA [1] have received much attention, due to its capability of handling more complicated design specifications and that the global optimal solutions can be obtained efficiently. Their applications in data-independent VWA beampattern synthesis for uniform linear array (ULA) were first introduced by Lebret *et al* [1]. However, the VWA based designs for non-uniform linear arrays are non-convex and hence are NP-hard [2]. Various beampattern synthesis methods based on VWA for adaptive arrays have been considered (see, e.g., [3]) for linear arrays and 2-D non-uniform arrays. However, a method that can solve the inherent problems with the standard Capon's method, such as varying main-beam shape, unmanageable peak sidelobe level (PSL), and sensitivity to the model errors, simultaneously according to prescribed parameters remains to be explored.

In this paper, we present optimal array pattern synthesis via Matrix Weighting Approaches (MWA), as shown in Fig. 1(b). The array geometry can be arbitrary. We focus on the array power response designs which are of interest in many applications including radar. Compared to the conventional VWA, weight matrices are used by MWA for much improved beampattern synthesis flexibility. For several different versions of MWA designs we have considered, globally optimal solutions can be determined efficiently due to convex optimization formulations. Numerical examples are presented to show the superior performance of MWA compared with their VWA counterparts.

2. PROBLEM FORMULATION

Consider an M-element array with an arbitrary array geometry. Let $s(n)$ denote the unknown waveform of a narrowband signal-of-interest (SOI), the received data model is given by [1]:

$$\mathbf{y}(n) = \mathbf{a}(\theta_0)s(n) + \mathbf{e}(n), \quad n = 1, \cdots, N, \tag{1}$$

where $\mathbf{y}(n)$ is the nth received data vector, $n = 1, \cdots, N$, with N denoting the snapshot number; $\mathbf{a}(\theta_0)$ is the array steering vector, θ_0 denote a generic source location parameter, and $\mathbf{e}(n)$ is the the noise and interference term.

In VWA, for a weight vector $\mathbf{w} = [w_1, \cdots, w_m]^T \in \mathcal{C}^{M\times 1}$, the power response of the array as a function of θ is the array beampattern [1]:

$$P(\theta) = |\mathbf{w}^*\mathbf{a}(\theta)|^2 = \mathbf{a}(\theta)^*\mathbf{w}\mathbf{w}^*\mathbf{a}(\theta) = \mathbf{a}(\theta)^*\mathbf{T}\mathbf{a}(\theta), \tag{2}$$

where the matrix $\mathbf{T} = \mathbf{w}\mathbf{w}^* \in \mathcal{C}^{M\times M}$ has rank one. In VWA design problems, the variable to be designed is the weight vector $\mathbf{w}$, or equivalently, the weight matrix $\mathbf{T}$ subject to the rank-one constraint. Due to the non-convexity of the rank-one constraint [4], the general VWA design problems cannot be solved in polynomial time and the solutions cannot be guaranteed to be globally optimal.

This work was sponsored by the Defense Advanced Research Projects Agency under Grant No. HR0011-06-1-0031 and under Air Force contract FA8721-05-C-0002. Opinions, interpretations, conclusions, and recommendations are those of the authors and are not necessarily endorsed by the United States Government. The second author performed this work while a Visiting Scientist at the MIT Lincoln Laboratory.

In array pattern synthesis via MWA, we remove the rank-1 constraint on $\mathbf{T}$. The resulting formulation is actually the Semidefinite Relaxation (SDR) [4] of the corresponding VWA formulation for the same beampattern synthesis problem. The goal of MWA is to find an optimal $\mathbf{T} \geq 0$ that gives a desired array beampattern. Once $\mathbf{T}$ is determined, we let $\mathbf{W} = \mathbf{\Sigma}^{1/2}\mathbf{U}$, where the columns of $\mathbf{U}$ are the eigenvectors of $\mathbf{T}$, the diagonal elements of $\mathbf{\Sigma}$ are the corresponding eigenvalues, and $\mathbf{W} = [\mathbf{w}_1, \cdots, \mathbf{w}_K] \in \mathcal{C}^{M\times K}$, with $\mathbf{w}_k$ denoting the kth weight vector in Fig. 1(b). Then the array beampattern of MWA is given by:

$$P(\theta) = \mathbf{a}^*(\theta)\mathbf{T}\mathbf{a}(\theta) = \mathbf{a}^*(\theta)\mathbf{W}\mathbf{W}^*\mathbf{a}(\theta) = \sum_{k=1}^{K} |\mathbf{a}^*(\theta)\mathbf{w}_k|^2 . \quad (3)$$

Therefore, MWA can be viewed as a filter-bank approach. We will consider two types of gain constraints : i) the total gain constraint, which requires tr$(\mathbf{T}) = c$, where c is some given constant (note that tr$(\mathbf{T}) = \sum_{k=1}^{K} \|\mathbf{w}_k\|^2$ and hence the total gain constraint is the trace constraint on $\mathbf{T}$); ii) the elemental uniform gain constraint, i.e., $T_{mm} = c/M$, $m = 1, \cdots, M$, which requires that each antenna element contributes equal gain to the array output. Both the constraints are linear in $\mathbf{T}$ and hence easy to solve. We observe from examples that the solved $\mathbf{T}$ is often low rank, which means that MWA will not require much higher hardware implementation cost than VWA.

3. DATA-ADAPTIVE MWA: AWESOME

Consider the data model in (1). Similar to the formulation of the Capon's beamformer [5], we aim at minimizing the array output power under the constraint of unit power gain for the SOI, and the constraints for the 3-dB main-beam width as well as the PSL. We referred to the data adaptive MWA as the Adaptive WEighting of Signals via One Matrix Entity (AWESOME), which is formulated as follows:

$$\begin{aligned}
&\min_{\mathbf{T}} && \mathrm{tr}(\hat{\mathbf{R}}\mathbf{T}) && (4)\\
&\text{subject to} && \mathbf{a}^*(\theta_0)\mathbf{T}\mathbf{a}(\theta_0) = 1, && (5)\\
& && \mathbf{a}^*(\theta_i)\mathbf{T}\mathbf{a}(\theta_i) = 0.5, \quad i = 1, 2, && (6)\\
& && \mathbf{a}^*(\mu_l)\mathbf{T}\mathbf{a}(\mu_l) \leq \varsigma, \quad \mu_l \in \Psi_s, && (7)\\
& && \mathbf{a}^*(\mu_l)\mathbf{T}\mathbf{a}(\mu_l) \geq 0.5, \quad \mu_l \in (\theta_1, \theta_2), && (8)\\
& && \mathbf{T} \geq 0. && (9)
\end{aligned}$$

where θ_0 is the location of the SOI, θ_1 and θ_2 are the prescribed 3-dB points, ς is the desired PSL, Ψ_s denotes the sidelobe region, the interval (θ_1, θ_2) is the 3-dB main-beam region, and $\hat{\mathbf{R}} = \frac{1}{N}\sum_{n=1}^{N} \mathbf{y}(n)\mathbf{y}^H(n)$ is the sample covariance matrix. The formulation in (4) is a Semi-Definite Program (SDP) [1] and can be solved efficiently via the SDP solvers such as SeDuMi [6]. The estimated SOI power $\hat{P}_s$ can be computed via $\hat{P}_s = \mathrm{tr}(\hat{\mathbf{R}}\mathbf{T})$.

4. DATA-INDEPENDENT MWA

4.1. Beampattern Matching Design

Beampattern matching design aims at finding the matrix $\mathbf{T} \geq 0$ such that $P(\theta)$ matches or rather approximates (in a mean-squared error (MSE) sense) the desired beampattern $P_d(\beta)$, over the region of interest Ω (covered by a fine grid of points $\{\mu_l\}_{l=1}^{L}$) under either the elemental uniform gain constraint or the total gain constraint. The formulation is given by:

$$\begin{aligned}
&\min_{\alpha,\mathbf{T}} && \frac{1}{L}\sum_{l=1}^{L} v_l \left[\alpha P_d(\mu_l) - \mathbf{a}^*(\mu_l)\mathbf{T}\mathbf{a}(\mu_l)\right]^2 && (10)\\
&\text{subject to} && T_{mm} = \frac{c}{M}, \quad m = 1, \cdots, M, && \\
& && \text{or} \quad \mathrm{tr}(\mathbf{T}) = c, && (11)\\
& && \mathbf{T} \geq 0, && (12)
\end{aligned}$$

where $v_l \geq 0$, $l = 1, \cdots, L$, is the weighting factor for the lth grid point. The scaling factor α is introduced since typically $\phi(\theta)$ is given in a "normalized form", and our interest lies in approximating an appropriately scaled version of $\phi(\theta)$, not $\phi(\theta)$ itself. By using techniques similar to those used in [7], we can reformulate the beampattern matching design in (10) as a Semi-definite Quadratic Programming (SQP) problem [1], which can be efficiently solved on a personal computer using public domain software such as SeDuMi [6].

4.2. Minimum Sidelobe Level Design

In some applications, it is important to control the PSL of the beampattern [3], while maintaining the shape of the main-lobe. For such purposes, we will consider the following minimum sidelobe level design:

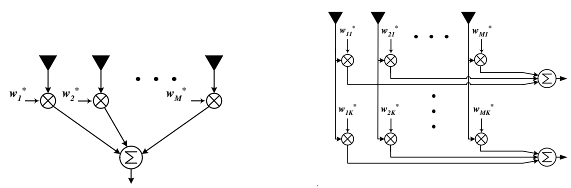
$$\begin{aligned}
&\min_{\mathbf{T}} \quad -t \quad \text{subject to} && (13)\\
&\mathbf{a}^*(\theta_0)\mathbf{T}\mathbf{a}(\theta_0) - \mathbf{a}^*(\mu_p)\mathbf{T}\mathbf{a}(\mu_p) > t, \quad \mu_p \in \Psi_s && (14)\\
&|\mathbf{a}^*(\theta_0)\mathbf{T}\mathbf{a}(\theta_0) - \mathbf{a}^*(\mu_p)\mathbf{T}\mathbf{a}(\mu_p)| \leq 0.5, \quad \mu_p \in \Psi_m && (15)\\
&0.5\mathbf{a}^*(\theta_0)\mathbf{T}\mathbf{a}(\theta_0) - \mathbf{a}^*(\theta_i)\mathbf{T}\mathbf{a}(\theta_i) = 0, \quad i = 1, 2, && (16)\\
&T_{mm} = c/M, \quad m = 1, \cdots, M, \quad \text{or} \quad \mathrm{tr}(\mathbf{T}) = c, && (17)\\
&\mathbf{T} \geq 0, && (18)
\end{aligned}$$

where the main-beam is directed toward θ_0, the prescribed 3-dB angles are θ_1 and θ_2, the 3-dB main-beam region is Ψ_m and the sidelobe region is Ψ_s. The formulation (13) is a SDP [1] and can be solved efficiently using SeDuMi [6].

The VMA counterparts can be readily modified from the previously described designs by adding the constraint rank$(\mathbf{T}) = 1$. However, due to the non-convexity of the rank constraint, the problem becomes much harder to solve and no globally optimal solution is guaranteed. In our numerical examples we have used the Newton-like algorithm [4] to find the rank-one solution.

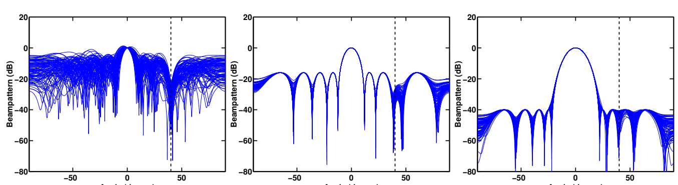
4.3. Constant Beamwidth Design for Wideband Arrays

Beamformers with frequency-invariant main-beam widths are desirable in many wideband signal processing applications, such as aeroacoustics [8] and communications. One way to achieve this goal is to apply frequency-dependent weights to the sensor outputs. However, there is no systematic way to design such shades, with various constraints, for non-uniform arrays such as the Small Aperture Directional Array (SADA) [8]. MWA can be readily applied to achieve constant beamwidth beampattern designs for wideband arrays. For example, we can use the minimum sidelobe level design of MWA by specifying a common 3-dB main-beam width for all frequency bins, and then obtaining a weight matrix for each narrowband frequency bin using (13). The resulting beampattern for each frequency bin will have a constant main-beam width and the lowest possible sidelobe level. The beampattern matching design of MWA and AWESOME can also be modified for constant beamwidth design of wideband arrays.



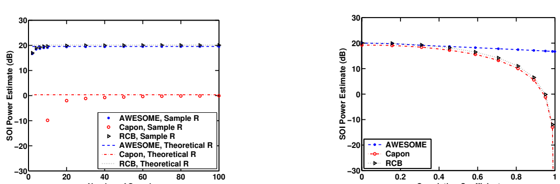
(a): VWA (b): MWA

Fig. 1. Diagrams for (a): VWA; (b): MWA.



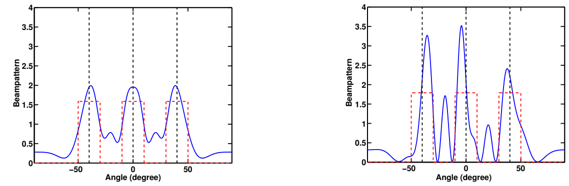
(a): Capon (b): MWA, -16 dB PSL (c): MWA, -40 dB PSL

Fig. 2. Beampatterns from 100 Monte Carlo trials via several adaptive beamforming methods.



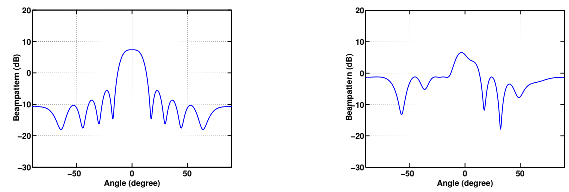
(a) (b)

Fig. 3. Power estimates for the SOI (a): versus N in the presence of a 2° steering angle error; (b): versus the correlation coefficient between the SOI and the interference.



(a) (b)

Fig. 4. Beampattern matching design for 5-element MRA.



(a) (b)

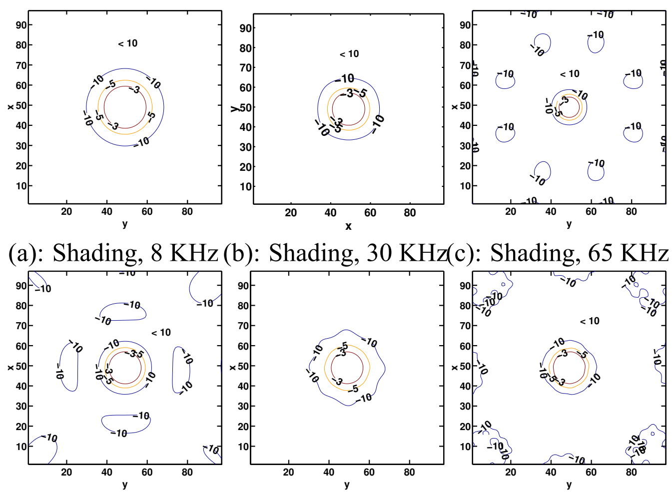
Fig. 5. Minimum sidelobe level design for 10-element ULA.

5. NUMERICAL EXAMPLES

First we study the capability of AWESOME for maintaining the main-beam shape and controlling the PSL. The beampatterns in Fig. 2 are obtained with 100 Monte-Carlo trials when $N = 50$. The SOI power is 20 dB. One strong interference with 60 dB power is present at $40°$. The data are simulated for a 10-element half-wavelength spacing ULA using (1), and the noise is assumed to be white complex Gaussian random process with zero-mean and covariance matrix $\mathbf{I}$. Fig. 2(a) corresponds to the standard Capon beamformer [5], where the PSL is as high as about -1 dB and the pointing location of the main-beam varies from trial to trial. Fig. 2(b) corresponds to AWESOME, with desired 3-dB points set to the same as the standard Capon beamformer in one trial (i.e., $-5.5°$ and $5°$). AWESOME can effectively control the PSL to below -16 dB, and maintain a constant main-beam shape from trial to trail. If the PSL must be as low as, say, -40 dB, we must broaden the desired 3-dB main-beam width to be between $-7.35°$ and $7.35°$ for AWESOME due to the trade off between the PSL and the main-beam width.

Second, we examine the robustness of AWESOME in the presence of small sample size and steering angle error problems. The assumed SOI angle is $0°$ while the true angle is $2°$. We change the interference power to 60 dB and all the other parameters are the same as Fig. 2(c). Note from Figs. 3(a), obtained from 100 Monte Carlo trails, that AWESOME and RCB [5] yield much more accurate SOI power estimates than the standard Capon beamformer.

The third example shows the performance of AWESOME when the interference is correlated with the SOI. We change the interference power to 50 dB, use the theoretical sample covariance matrix $\mathbf{R}$, and keep all the other parameters same as Fig. 2(c). AWESOME significantly outperforms both RCB



(a): Shading, 8 KHz (b): Shading, 30 KHz (c): Shading, 65 KHz

(d): MWA, 8 KHz (e): MWA, 30 KHz (f): MWA, 65 KHz

Fig. 6. Beampatterns for SADA at various frequencies (8, 30, 65 KHz).

[5] and the standard Capon beamformer, as shown in Fig. 3(b), obtained from 100 Monte Carlo trails, since the latter two algorithms fail to function properly and AWESOME has strict main-beam shape control.

Next, we consider the beampattern matching design in (10) with the the 5-element Minimum Redundancy Array (MRA), which is a non-uniform linear array with the same physical aperture as that of the 10-element ULA, under the elemental uniform gain constraint with $c = 1$. The desired beampattern has three pulses centered at $-40°$, $0°$, and $40°$, each with a width of $20°$. MWA can provide a much better beampattern matching than VWA, as shown in Fig. 4. The red line corresponds to the scaled desired beampattern.

Now we consider the minimum sidelobe level design in (13) for the 10-element ULA. The main-beam is centered at $0°$ with $\theta_1 = -10°$, $\theta_2 = 10°$, and $\Psi_s = [-90°, -20°] \cup [20°, 90°]$. Note from (5) that VWA fails to produce a proper main-beam and that the PSL of the VWA beampattern is more than 5 dB higher than that of MWA.

The last example shows the wideband constant beamwidth design for SADA [8], which is used for aeroacoustic noise measurement. The fact that SADA is a non-uniform 2-D array and that it is for near-field noise power measurements makes the problem more challenging. Beampatterns are obtained using the minimum sidelobe level design (13) under the total gain constraint. The 2-D beampatterns in Fig. 6 are the contours of the 3-D beampatterns on a plane parallel to and 4 feet above the array. The radius of the desired 3-dB contour for MWA is 4 inches. The beampatterns in Figs. 6(a) – 6(c) are obtained by using VMA shaded by the frequency-dependent weight vector, which is designed by trail-and-error [8] to maintain a constant beamwidth within the frequency band of 10 – 40 KHz. Note from Fig. 6 MWA is capable of extending this constant beamwidth working frequency band to 8 – 65 KHz.

6. CONCLUSIONS

We have presented a new approach for optimal array pattern synthesis for an arbitrary array. By deploying a weighting matrix (instead of a vector) at the array output, much improved flexibility for optimal array pattern synthesis can be achieved, and globally optimal solutions can be determined efficiently due to convex optimization formulations. Numerical examples have been demonstrated that the data-adaptive AWESOME allows for strict controls of main-beam shape and peak sidelobe level while retaining the capability of adaptive nulling of strong interferences and jammers, that AWESOME is robust against various modelling error, and that data-independent MWA can achieve much better beampattern synthesis compared to the VWA counterpart.

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