METHODS AND BOUNDS FOR ANTENNA ARRAY COUPLING MATRIX ESTIMATION

Marc Mowlér, Erik G. Larsson, Björn Lindmark, and Björn Ottersten

Royal Institute of Technology (KTH), EE, Signal Processing/Communication Theory
Osquldas väg 10, 100 44 Stockholm, Sweden. Email: marcm@ee.kth.se

ABSTRACT

A novel method is proposed for estimation of the mutual coupling matrix of an antenna array. The method extends previous work by incorporating an unknown phase center and the element factor (antenna radiation pattern) in the model, and treating these as nuisance parameters during the estimation of coupling. To facilitate this, a parametrization of the element factor based on a truncated Fourier series is proposed. The Cramér-Rao bound (CRB) for the estimation problem is derived and used to analyze how the required amount of measurement data increases when introducing a more and more flexible model for the element factor. Finally, the performance of the proposed estimator is illustrated using data from measurements on an 8-element antenna array.

Index Terms— Antenna array mutual coupling, Estimation, Fourier series, Cramér-Rao Lower Bound

1. INTRODUCTION

Adaptive antenna arrays in mobile communications introduce the possibility to increase the signal-to-noise ratio and suppress interference. This typically requires signal processing schemes that rely on specific assumptions on the array patterns. Commonly, the array is assumed ideal leading to reduced performance in practice [1–3]. The non-ideal behavior in reality is mainly due to the mutual coupling between the antenna elements. This coupling can be modelled via a coupling matrix whose inverse can be used to compensate the received data. In many cases, the coupling matrix can be obtained from electromagnetic simulations or raw calibration measurements [4–6]. In [2], such a compensation was found superior to other alternatives for dealing with coupling (such as using dummy columns), for the case of a 4-column dual polarized array.

Estimation of the coupling matrix from calibration measurements is a difficult problem, as many non-idealities and unknown factors tend to influence the accuracy one can achieve. For example, imperfect knowledge of (and incorrect compensation for) the element factors and the antenna phase center location during calibration measurements has shown to affect the estimation of the coupling matrix as well as the overall performance [7]. In this paper, we target these problems. Specifically, we present

- a novel method for *joint* estimation of the mutual coupling matrix, the element factor (radiation pattern), and the phase center of an antenna array (Sections 2-4),
- a CRB analysis which is used to motivate the model and which can be used as a performance benchmark as well (Section 5), and
- an illustration of the estimation performance based on measurements on an 8-element antenna array (Section 6).

Our paper extends previous work [4–6] in several directions. Most importantly, by treating the radiation pattern and array phase center as unknowns during the coupling estimation, we obtain a robust and versatile method for coupling matrix estimation without requiring the user to provide any *a priori* knowledge neither of the location of the array center nor about the individual antenna elements.

2. DATA MODEL AND PROBLEM FORMULATION

Consider a uniform linear antenna array with M elements and element-spacing d, receiving a signal emitted by a point source with direction-of-arrival θ relative to the array broadside. We will model the impulse response of this source as

$$\mathbf{h}(\mathbf{C}, \delta_x, \delta_y, f(\theta)) = \mathbf{C}\,\boldsymbol{a}(\theta)\, e^{jk(\delta_x \cos\theta + \delta_y \sin\theta)}\, f(\theta), \quad \mathbf{h} \in C^{M\times 1}$$

where

- $\mathbf{C}$ is the $M \times M$ coupling matrix. This matrix is complex-valued and unstructured.
- $\{\delta_x, \delta_y\}$ is the array phase center.
- $f(\theta) \geq 0$ is the element factor (antenna radiation pattern) in the direction θ. The function $f(\theta)$ is real-valued describing the amount of power radiated by the antenna in different directions θ, i.e., the radiation pattern includes no direction-dependent phase shift [8].
- $\boldsymbol{a}(\theta)$ is a Vandermonde vector where the mth element is
$$a_m(\theta) = e^{jkd\sin\theta(m-M/2-1/2)}$$
- d is the distance between the antenna elements.
- k is the wave number.

An implicit assumption in this model is that all antenna elements have the same radiation patterns, although this pattern in general is non-isotropic. (For a dipole, for instance, it is cosine-like.)

The objective is to estimate $\mathbf{C}$, using N independent measurements on point sources based on calibration data collected at known direction-of-arrivals[1] $\{\theta_1, \ldots, \theta_N\}$, when δ_x, δ_y and $f(\theta)$ are *unknown*. To this end we assume the user has measured one data vector $\mathbf{x}_n$ for each angle θ_n. By arranging these data in a matrix, $\mathbf{X} = [\mathbf{x}_1 \ \ldots \ \mathbf{x}_N]$, we obtain

$$\mathbf{X} = \mathbf{CAD}(\delta_x, \delta_y)\mathbf{E}(f(\theta)) + \mathbf{W} \tag{1}$$

$$\mathbf{A} = [\boldsymbol{a}(\theta_1) \ \cdots \ \boldsymbol{a}(\theta_N)], \quad \mathbf{E}(f(\theta)) = \mathrm{diag}\{f(\theta_1), \ldots, f(\theta_N)\}$$
$$\mathbf{D}(\delta_x, \delta_y) = \mathrm{diag}\{e^{jk(\delta_x\cos\theta_1 + \delta_y\sin\theta_1)}, \ldots, e^{jk(\delta_x\cos\theta_N + \delta_y\sin\theta_N)}\}$$

The matrix $\mathbf{W}$ represents measurement noise, which we will assume to be i.i.d. zero-mean complex Gaussian with variance σ^2 per component.

[1] Hereby the orientation of the array is assumed perfectly known, while the exact position of the phase center is typically unknown without thorough examination of the antenna performance.

3. PARAMETRIZATION OF THE ELEMENT FACTOR

The main difficulty at this point is that the element factor $f(\theta)$ is an unknown function. To be able to estimate it we need a parametrization. We have chosen to model $f(\theta)$ as a linear combination of sinusoidal basis functions:

$$f(\theta) = \sum_{k=1}^{K} \alpha_k \cos{(k-1)\theta}, \quad |\theta| < \pi/2 \tag{2}$$

where K is a known (small) integer, and $\boldsymbol{\alpha}$ are unknown, real-valued constants. Note that (2) is effectively equivalent to a truncated Fourier series and can assume negative values. We stress however that the reason for introducing the above is to increase the possibility to estimate $\mathbf{C}$ more accurately by allowing $\mathbf{E}$ to deviate from an omnidirectional pattern. While there are many possible alternative parameterizations of $f(\theta)$ (for example, one could use a piecewise constant function of θ), we find the one in (2) particularly attractive for the following reasons: The basis functions are orthogonal; the parametrization guarantees that $f(\theta)$ is smooth; and the parameters $\boldsymbol{\alpha}$ enter the model linearly. Additionally, Fourier expansions seem to have some general appeal in the context of modeling unknown functions. Under (2), we have

$$\mathbf{E} = \sum_{k=1}^{K} \alpha_k \boldsymbol{Q}_k, \quad \boldsymbol{\alpha} = [\alpha_1 \dots \alpha_K]^T$$
$$\boldsymbol{Q}_k = \text{diag}\{\cos{(k-1)\theta_1}, \dots, \cos{(k-1)\theta_N}\} \tag{3}$$

4. ESTIMATION OF THE COUPLING MATRIX

We propose to estimate $\mathbf{C}$, $\boldsymbol{\alpha}$, δ_x, and δ_y from $\boldsymbol{X}$ by using a least-squares criterion:[2]

$$\min_{\mathbf{C},\boldsymbol{\alpha},\delta_x,\delta_y} \|\mathbf{X} - \mathbf{CAD}(\delta_x, \delta_y)\mathbf{E}(\boldsymbol{\alpha})\|_F^2 \tag{4}$$

This is equivalent to maximum-likelihood estimation since the noise is assumed Gaussian. To find the minimizing $\mathbf{C}, \boldsymbol{\alpha}, \delta_x, \delta_y$, we iterate the following three steps:

1. Minimize (4) with respect to $\mathbf{C}$ while δ_x, δ_y, and $\boldsymbol{\alpha}$ fixed:

$$\hat{\mathbf{C}} = \mathbf{X}(\mathbf{ADE})^H[\mathbf{ADE}(\mathbf{ADE})^H]^{-1} \tag{5}$$

2. Minimize (4) with respect to $\boldsymbol{\alpha}$ while $\mathbf{C}$, δ_x, and δ_y fixed:

$$\hat{\boldsymbol{\alpha}} = (\mathbf{Y}^T\mathbf{Y})^{-1}\mathbf{Y}^T\mathbf{x}, \quad \mathbf{x} = \begin{bmatrix} \text{vec}\{\text{Re}\{\mathbf{X}\}\} \\ \text{vec}\{\text{Im}\{\mathbf{X}\}\} \end{bmatrix} \tag{6}$$

$$\mathbf{Y} = \begin{bmatrix} \text{vec}\{\text{Re}\{\mathbf{CADQ}_1\}\} & \dots & \text{vec}\{\text{Re}\{\mathbf{CADQ}_K\}\} \\ \text{vec}\{\text{Im}\{\mathbf{CADQ}_1\}\} & \dots & \text{vec}\{\text{Im}\{\mathbf{CADQ}_K\}\} \end{bmatrix}$$

3. Minimize (4) with respect to δ_x and δ_y while keeping $\mathbf{C}$ and $\boldsymbol{\alpha}$ fixed via a two-dimensional gradient search.

In the first iteration we must provide the algorithm with an initial estimate. Initially, let $\mathbf{D} = \mathbf{I}$ and $\boldsymbol{\alpha} = [0, 1, 0, \dots, 0]^T$ which corresponds to the element factor $f(\theta) = \cos\theta$ which was used in [6].

[2] The minimum of this cost function is not unique, see Section 5 for a discussion of this.

5. ANALYSIS OF FEASIBLE CHOICE OF K

One of the most important questions regarding the model of Section 2 is what value of K (i.e., level of flexibility of the element factor) is feasible. Clearly the larger K one uses, the more accurately an arbitrary element factor can be represented; however, at the same time the more measurements (N) must be acquired, or a correspondingly larger signal-to-noise-ratio [SNR] must be arranged for. In this section we attempt to quantify how the choice of K will affect the required value of N and SNR, $1/\sigma^2$. For this analysis we derive the Cramér-Rao bound (CRB) for the estimation problem in (4).

To obtain the CRB, we collect all (real-valued) unknowns into the vector

$$\boldsymbol{\xi} = \quad [C_{11}^R ... C_{1M}^R \quad ... C_{MM}^R \quad ... C_{ij}^I \quad \delta_x \quad \delta_y \quad \alpha_1 ... \alpha_K]^T$$

The CRB for the estimation of $\boldsymbol{\xi}$ is then given by [9]:

$$\mathbf{CRB} = [\mathbf{I}_{\text{Fisher}}]^{-1}, \quad [\mathbf{I}_{\text{Fisher}}]_{ij} = \frac{2}{\sigma^2}\text{Re}\left\{\frac{\partial \boldsymbol{\mu}^H}{\partial \xi_i}\frac{\partial \boldsymbol{\mu}}{\partial \xi_j}\right\} \tag{7}$$

where $\boldsymbol{\mu}$ is the expected value of $\mathbf{x}$ in (6).

Note that the problem (4) is unidentifiable as it stands: there is a scaling ambiguity between $\mathbf{C}$ and α. Therefore a constraint is needed on the problem. We choose to constrain $\|\mathbf{C}\|_F^2 = M$. Other possible constraints (like $\alpha_1 = 1$ or $C_{11}^R = 1$ for example) favor particular elements of $\mathbf{C}$ or α, which is not desirable. This was also implemented as a normalization in the estimator presented in the previous section, although not explicitly mentioned there. A general formula for the CRB under parametric constraints was derived in [10]. By writing the constraint as

$$g(\boldsymbol{\xi}) = \|\mathbf{C}\|_F^2 - M = \text{Tr}\{\mathbf{C}^H\mathbf{C}\} - M = 0 \tag{8}$$

and defining

$$[\mathbf{G}(\boldsymbol{\xi})]_l = \frac{\partial g(\boldsymbol{\xi})}{\partial \xi_l} = \begin{cases} 2\text{Re}\{c_{ij}\} & \text{if } \xi_l = \text{Re}\{c_{ij}\} \\ 2\text{Im}\{c_{ij}\} & \text{if } \xi_l = \text{Im}\{c_{ij}\} \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

the constrained CRB is given by [10]

$$\mathbf{CRB} = \mathbf{U}(\mathbf{U}^T\mathbf{I}_{\text{Fisher}}\mathbf{U})^{-1}\mathbf{U}^T \tag{10}$$

where $\mathbf{U}$ is implicitly defined via $\mathbf{G}(\boldsymbol{\xi})\mathbf{U} = \mathbf{0}$. Note that the CRB is proportional to σ^2, i.e., the estimation accuracy is inversely proportional to the SNR.

The CRB as a function of K (i.e., the order of the parametrization of $f(\theta)$) for an 8-element array of dipoles over ground plane is presented in Figure 1. When generating this figure, we used the theoretical models of [8] for the coupling between antenna elements. The true data used are based on a scenario when $K = 1$, $\mathbf{D} = \mathbf{I}$, and $\mathbf{E} = \mathbf{I}$. The SNR was 0 dB ($\sigma^2 = 1$). The four curves show

- $\text{CRB}_C = \text{Tr}\{\{\mathbf{U}_C(\mathbf{U}_C^T \text{I}_C \mathbf{U}_C)^{-1}\mathbf{U}_C^T\}_C\}$, the CRB of $\mathbf{C}$ when both $\mathbf{D}$ and $\mathbf{E}$ are known
- $\text{CRB}_{CD} = \text{Tr}\{\{\mathbf{U}_{CD}(\mathbf{U}_{CD}^T \text{I}_{CD} \mathbf{U}_{CD})^{-1}\mathbf{U}_{CD}^T\}_C\}$, the CRB of $\mathbf{C}$ when $\mathbf{D}$ is unknown and $\mathbf{E}$ is known
- $\text{CRB}_{CE} = \text{Tr}\{\{\mathbf{U}_{CE}(\mathbf{U}_{CE}^T \text{I}_{CE} \mathbf{U}_{CE})^{-1}\mathbf{U}_{CE}^T\}_C\}$, the CRB of $\mathbf{C}$ when $\mathbf{D}$ is known and $\mathbf{E}$ is unknown
- $\text{CRB}_{CDE} = \text{Tr}\{\{\mathbf{U}_{CDE}(\mathbf{U}_{CDE}^T \text{I}_{CDE} \mathbf{U}_{CDE})^{-1}\mathbf{U}_{CDE}^T\}_C\}$, the CRB of $\mathbf{C}$ when $\mathbf{D}$ and $\mathbf{E}$ are unknown

The results in Figure 1 quantify the increase in achievable estimation performance for the elements of $\mathbf{C}$ when increasing the number of nuisance parameters in the model. In particular we see that the estimation problem becomes more difficult when more and more $\boldsymbol{\alpha}$ parameters are introduced in the model. For example, it is more difficult to estimate the coupling matrix $\mathbf{C}$ when the phase center δ_x, δ_y, is unknown, compared to when it is not. However, for $K \geq 3$, the difficulty of identifying $\boldsymbol{\alpha}$ dominates over the problem of estimating the phase center. Since the CRB is proportional to σ^2, the increase in emitter power (i.e., SNR) needed to maintain a given performance when the model is expanded with more unknowns can be directly read out from the figure.

In Figure 2 we study the CRB as a function of K and N. From this figure we can directly read out how much higher emitter power (or equivalently, lower σ^2) is required to be able maintain the same estimation performance for $\mathbf{C}$ when K or N vary. For example, we can see that if we fix $N = 100$, say, then going from $K = 1$ to $K = 2$ requires 1 dB extra SNR. Going from $K = 2$ to $K = 3$ needs an increase of the SNR level by 1.5 dB. However going from $K = 3$ to 4 requires 10 dB extra SNR. Thus, $K = 3$ seems to be a reasonable choice. In practice, it is difficult to handle a more flexible model for the element factor.

In Figure 2 we also see that in the limit where the number of angles reach $N = 15$, the problem becomes unidentifiable. Note that for $N \leq 15$ the number of unknown parameters in the model exceeds the number of recorded samples. From Figure 2 we can also make many other interesting observations. For instance, we see that for fixed K increasing N from 100 to 200 is approximately equivalent to increasing the SNR with 3 dB. This holds in general: For $N \gg 1$, doubling the SNR gives the same effect as doubling N.

6. EVALUATION OF ESTIMATOR BASED ON MEASUREMENTS

Data from an 8-column antenna array were collected during calibration measurements with 180 measurement points spread out over a half-circle. The estimator presented in this paper was used to estimate the coupling matrix of this antenna array. The estimated coupling matrix was then used to precompensate the data, after which radiation patterns can be obtained. Radiation patterns and phase errors before and after compensation are presented in Figure 3 for measured data.

Figure 3(a) shows the individual radiation patterns of each antenna element as measured during calibration ($\mathbf{X}$). Figure 3(b) shows the radiation patterns after compensation by the coupling matrix ($\mathbf{C}^{-1}\mathbf{X}$) when isotropic conditions are assumed by the estimator. This means assuming $\mathbf{E} = \mathbf{I}$, which is equivalent to setting $K = 1$ in our algorithm. The radiation patterns after compensation by the coupling matrix when using the proposed estimator with $K = 3$ are presented in Figure 3(c).

The results using an isotropic assumption on the element factor, Figure 3(b), shows an improvement over the uncompensated data of Figure 3(a), but it is still inferior to the case when a more sophisticated model is assumed for the element factor (see Figure 3(c)). Using $K = 3$ gives a result closer to the ideal (without coupling) array response compared to the case when isotropic element factors are assumed during the estimation of $\mathbf{C}$.

Phase diagrams representing the average phase error of the coupling matrix before and after compensation with the coupling matrix are presented in Figure 3(d). The phase error after compensation with $K = 3$ (bottom), modelling the phase shift and the element factor, is less than without the compensation (top). Assuming an isotropic element factor (middle) gives a better result than without compensation but worse than the result of our method. This indicates that the validity of the estimated coupling matrix based on phase considerations increases with the proposed estimator.

7. CONCLUSIONS

We introduced a new method for the estimation of the mutual coupling matrix of an antenna array. The main novelty over existing methods was that the array phase center and the element factors were introduced as unknowns in the data model and treated as nuisance parameters in the estimation of the coupling.

Most of our quantitative studies focused on an 8-element linear array. For this setup some specific conclusions were:

- The SNR penalty associated with introducing a model for the element factor with two degrees of freedom ($K = 3$) was 2.5 dB. This means that an additional 2.5 dB more power (or a doubling of the number of accumulated samples) must be used to retain the estimation accuracy of the coupling matrix compared to the case when the algorithm assumed omnidirectional elements. To add another degree of freedom (set $K = 4$) costs another 10 dB.
- In an experimental study using measured calibration data we found that the proposed method and the associated estimator could significantly improve the quality of the estimated coupling matrix, and the result of subsequent compensation processing.

8. REFERENCES

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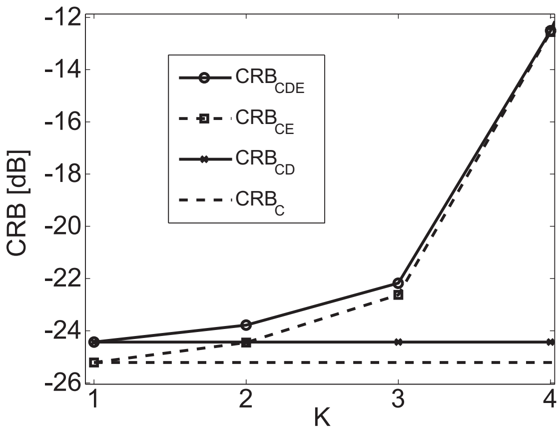


Fig. 1. CRB for the elements of $\mathbf{C}$ under different assumptions on whether $\mathbf{D}, \mathbf{E}$ were known or not, and for different K.

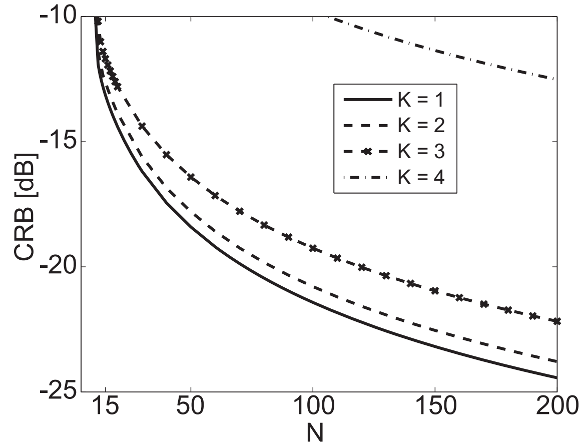
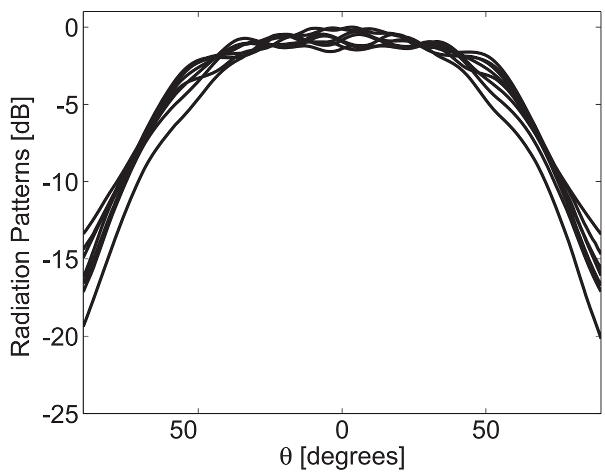
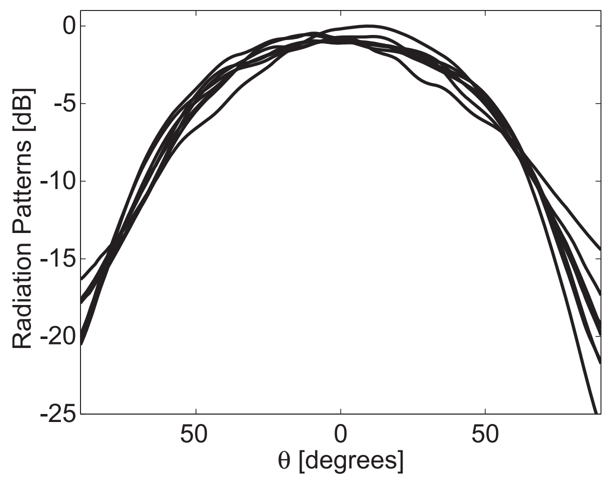


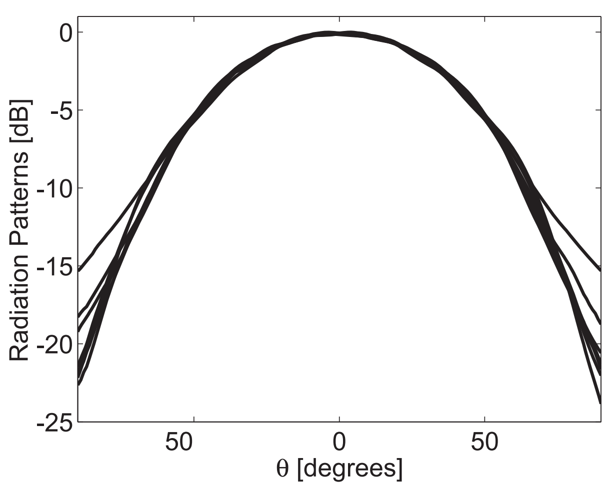
Fig. 2. CRB for the elements of $\mathbf{C}$ as a function of K and N when SNR is 0 dB.



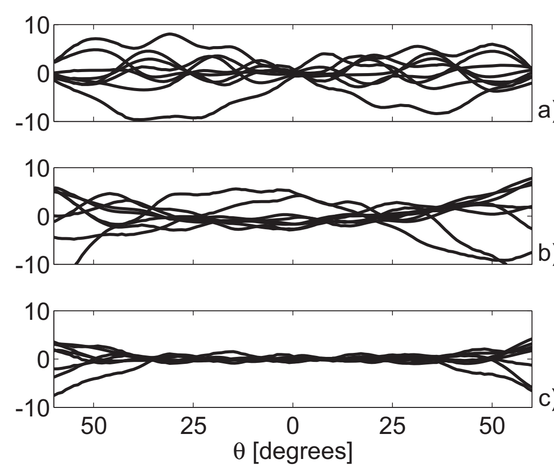
(a) Radiation patterns obtained from *uncompensated* array data.



(b) Radiation patterns obtained from *compensated* data using a $\mathbf{C}$ matrix estimated via our algorithm setting $K = 1$ (i.e., forcing $\mathbf{E} \propto \mathbf{I}$).



(c) *Compensated* radiation patterns using our method with $K = 3$.



(d) Phase errors for the cases in (a), (b), and (c).

Fig. 3. Radiation patterns obtained from compensated data with the coupling matrix $\mathbf{C}$ estimated in different ways, based on measurements from an 8-element array.