A 2×2 SPACE-TIME CODE WITH NON-VANISHING DETERMINANTS AND FAST MAXIMUM LIKELIHOOD DECODING

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ABSTRACT

A new 2×2 full-rate full-diversity space-time block code (STBC) is proposed that satisfies the non-vanishing determinant property and offers a reduced computational complexity as compared to the other existing full-rate codes. The performance of our new STBC is shown to be comparable to that of the best full-rate STBCs known so far. This performance is achieved at the decoding complexity which is substantially lower than that of the standard sphere decoder.

Index Terms— Full-rate full-diversity space-time codes, non-vanishing determinant property, sphere decoder

1. INTRODUCTION

Space-time coding has emerged as an important approach to improve the performance and capacity of multiple-input multiple-output (MIMO) communication systems. Space-time codes efficiently exploit the spatial diversity offered by the use of multiple antennas at the transmitter and/or receiver.

Different approaches have been used to design STBC techniques. The orthogonal STBCs (OSTBCs) proposed in [1] and [2] achieve full diversity at a low maximum likelihood (ML) decoding complexity, but their achievable rate is limited by the code orthogonality property. Full rate is achieved by the STBCs of [3] but with no full diversity guarantee. The vertical Bell Labs layered space-time (V-BLAST) scheme of [4] offers full rate and simple decoding complexity but does not provide any diversity gain. Recent approaches to designing powerful STBCs have used number theory tools to develop space-time codes that have full rate and full diversity [5]-[8]. Unfortunately, the ML decoder complexity for these codes may be rather high as they are based on sphere decoding. The use of sub-optimal decoding schemes can achieve a lower complexity but at the price of a significant reduction in the performance.

In this paper, we address the problem of designing full-rate STBCs that offer a reasonable tradeoff between the performance and decoding complexity. A new 2×2 STBC is proposed that exploits properties of the OSTBC generator matrix to reduce the decoding complexity which lies for the proposed code in between that of the symbol-by-symbol decoder and the standard sphere decoder (which is used by all full-rate space-time codes). Our code is shown to have non-vanishing determinants and, therefore, achieves the diversity-multiplexing gain tradeoff. Its performance is shown to be comparable to the best full-rate STBCs known so far.

2. SYSTEM MODEL

Let us consider a Rayleigh quasi-static flat fading MIMO channel with N_t transmit and N_r receive antennas. The $N_r \times T$ received data block can be modeled as [2]

$$\mathbf{Y} = \sqrt{\frac{\mathsf{SNR}}{N_t}}\mathbf{H}\mathbf{X} + \mathbf{N} \tag{1}$$

where $\mathbf{X}$ is the $N_t \times T$ code matrix, $\mathbf{H}$ and $\mathbf{N}$ are the $N_r \times N_t$ channel matrix and the $N_r \times T$ noise matrix, respectively, whose entries are i.i.d. complex random variables with the pdf $\mathcal{CN}(0,1)$, SNR is the average signal-to-noise ratio at each receive antenna, and T is the transmitted block length. We further assume that the receiver has a perfect channel state information (CSI), and that the K symbols s_k, $k = 1 \ldots K$ drawn from the M-QAM constellation are encoded to form the matrix $\mathbf{X}$ as [3]

$$\mathbf{X} = \sum_{k=1}^{K} (s_{rk}\mathbf{C}_k + s_{ik}\mathbf{D}_k) \tag{2}$$

where s_{rk} and s_{ik} are the real and imaginary parts of s_k, respectively, $\{\mathbf{C}_k\}_{k=1}^K$ and $\{\mathbf{D}_k\}_{k=1}^K$ are two sets of complex $N_t \times T$ matrices that have to be designed subject to the following constraint

$$\sum_{k=1}^{K} \mathrm{tr}\left(\mathbf{C}_k^H\mathbf{C}_k + \mathbf{D}_k^H\mathbf{D}_k\right) = 2TN_t \tag{3}$$

$(\cdot)^H$ denotes the conjugate transpose, and tr$(\cdot)$ stands for the trace of a matrix. The codebook produced by encoding the symbols s_k can be defined as

$$\mathcal{X} \triangleq \{\mathbf{X}_1, \ldots, \mathbf{X}_L\}$$

where the cardinality of the codebook is $L = M^K$. Therefore the rate of transmission is $R = \frac{\log_2 L}{T}$ bits per channel use (bcu).

For any $I \times J$ matrix $\mathbf{Z}$, let us define the "underline" operator which transforms this matrix into a $2IJ \times 1$ real column vector as follows

$$\underline{\mathbf{Z}} \triangleq [\mathrm{Re}\{Z_{11}\}, \mathrm{Im}\{Z_{11}\}, \mathrm{Re}\{Z_{21}\}, \mathrm{Im}\{Z_{21}\}, \ldots, \mathrm{Re}\{Z_{IJ}\}, \mathrm{Im}\{Z_{IJ}\}]^T. \quad (4)$$

Using (4), it can be shown [9] that (1) can be written as

$$\underline{\mathbf{Y}} = \sqrt{\frac{\mathsf{SNR}}{N_t}} \mathbb{H} \underline{\mathbf{X}} + \underline{\mathbf{N}} \quad (5)$$

where $\mathbb{H} = \frac{1}{2} \mathbf{I}_T \otimes (\mathbf{H} \otimes \mathbf{E} + \mathbf{H}^* \otimes \mathbf{E}^*)$, $\mathbf{E} = \begin{bmatrix} 1 & j \\ -j & 1 \end{bmatrix}$, $j = \sqrt{-1}$, $\mathbf{I}_T$ is the $T \times T$ identity matrix, $\otimes$ denotes the Kronecker product, and $(\cdot)^*$ stands for the complex conjugate.

Using (2) in (5), we have

$$\underline{\mathbf{X}} = \mathbb{G} \underline{\mathbf{s}} \quad (6)$$

where the $2N_tT \times 2K$ real matrix

$$\mathbb{G} \triangleq [\, \underline{\mathbf{C}_1} \; \underline{\mathbf{D}_1} \cdots \underline{\mathbf{C}_K} \; \underline{\mathbf{D}_K} \,] \quad (7)$$

is the code generator matrix, and $\underline{\mathbf{s}}$ is the underline version of the symbol vector $\mathbf{s} = [s_1 \ldots s_K]^T$. The constraint in (3) is now equivalent to

$$\mathrm{tr}(\mathbb{G}^T \mathbb{G}) = 2TN_t. \quad (8)$$

If $N_rT \geq K$ and $\mathbb{H}$ is full-rank, then the coherent ML decoder is given by

$$\hat{\underline{\mathbf{s}}} = \arg\min_{\underline{\mathbf{s}}_l} \|\underline{\mathbf{Y}} - \mathbb{H}\mathbb{G}\underline{\mathbf{s}}_l\| \quad (9)$$

where $\|\cdot\|$ is the Euclidean norm.

The latter decoder can be implemented using the sphere decoding algorithm [10] which in most cases is much more computationally efficient than the exhaustive search.

We will call a lattice $\underline{\mathcal{X}} = \{\underline{\mathbf{X}}_1, \ldots, \underline{\mathbf{X}}_L\}$ orthogonal if the columns of the corresponding generator matrix $\mathbb{G}$ are orthogonal. Note that for the lattice of $\underline{\mathcal{Y}} = \{\underline{\mathbf{Y}}_1, \ldots, \underline{\mathbf{Y}}_L\}$ to be orthogonal, it is not sufficient that $\underline{\mathcal{X}}$ is orthogonal because of the effect of randomness in the channel.

3. THE NEW SPACE-TIME CODE

Let us briefly discuss the OSTBCs first. Any $\mathbf{X}$ is said to be an OSTBC [2] if it is a linear combination of the K entries s_k and

$$\mathbf{X}\mathbf{X}^H = \|\mathbf{s}\|^2 \mathbf{I}_{N_t}. \quad (10)$$

According to the constellation space invariance property of OSTBCs [9], the orthogonality of the lattice remains invariant to the skewing effects of the channel matrix, guaranteeing that the coherent ML decoder can be implemented as a simple symbol-by-symbol decoder.

Let us hereafter consider the case of $N_t = T = 2$ and $K = N_tT = 4$ so that the code has full rate at which no OSTBC exists. To design such a code, we have to choose $2N_tT = 2K$ linearly independent columns of $\mathbb{G}$. To maintain the full rate restriction but, at the same time, to simplify the complexity of the associated ML decoder, let us choose as many first columns of $\mathbb{G}$ as possible from an OSTBC (in our particular 2×2 case from the Alamouti's code [1]). By doing so, we guarantee that, according to the constellation space invariance property, the corresponding axes of the lattice $\underline{\mathcal{Y}} = \{\underline{\mathbf{Y}}_1, \ldots, \underline{\mathbf{Y}}_L\}$ are orthogonal at the decoder. The remaining columns of $\mathbb{G}$ are selected to be orthogonal to the first OSTBC-based columns. Additionally, we require the matrix $\mathbb{G}$ to be orthogonal, i.e., $\mathbb{G}^T\mathbb{G} = \mathbb{G}\mathbb{G}^T = \mathbf{I}_{2\mathrm{K}}$. In [6], it was shown that this property is a sufficient and necessary condition for the code to be information-lossless. It can be readily shown that if the last $K = 4$ columns $\mathbf{g}_k$, $k = 1, 2, 3, 4$ of the matrix

$$\mathbb{G} = [\mathbb{G}_{\mathrm{OSTBC}} \; \mathbf{g}_1 \; \mathbf{g}_2 \; \mathbf{g}_3 \; \mathbf{g}_4] \quad (11)$$

are chosen as

$$\mathbf{g}_k = [g_{1k}, g_{2k}, g_{3k}, g_{4k}, g_{3k}, -g_{4k}, -g_{1k}, g_{2k}]^T \quad (12)$$

then for any g_{lk} ($l = 1, 2, 3, 4;\ k = 1, 2, 3, 4$) they are orthogonal to the first K OSTBC-based columns, where $\mathbb{G}_{\mathrm{OSTBC}}$ is the generator matrix of the Alamouti's code.

Choosing the elements of the vectors $\mathbf{g}_k$ ($k = 1, 2, 3, 4$), we have to satisfy the following property:

$$\mathbf{g}_k^T \mathbf{g}_l = \begin{cases} 1 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases} \quad (13)$$

for all $k = 1, 2, 3, 4$ and $l = 1, 2, 3, 4$. According to (12), the latter property is equivalent to

$$\tilde{\mathbf{g}}_k^T \tilde{\mathbf{g}}_l = \begin{cases} 1/2 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases} \quad (14)$$

where the 4×1 vector $\tilde{\mathbf{g}}_k$ contains the four upper entries of $\mathbf{g}_k$. We can satisfy (14) by introducing a 4×4 matrix

$$\tilde{\mathbb{G}} \triangleq \sqrt{2}\,[\tilde{\mathbf{g}}_1 \; \tilde{\mathbf{g}}_2 \; \tilde{\mathbf{g}}_3 \; \tilde{\mathbf{g}}_4]. \quad (15)$$

Note that $\tilde{\mathbb{G}}$ is an orthogonal matrix and, therefore, it can be parameterized using Givens rotations in the way proposed in [11]. This gives us six degrees of freedom which can be used to satisfy some desired performance criterion.

We have optimized our code using the well-known criterion that, to achieve full diversity, all non-zero codeword difference matrices must be full rank [12]. By maximizing the absolute value of the worst codeword-difference matrix determinant

$$\min_{\substack{\mathbf{X}, \mathbf{X}' \in \mathcal{X} \\ \mathbf{X} \neq \mathbf{X}'}} \delta(\tilde{\mathbb{G}}), \qquad \delta(\tilde{\mathbb{G}}) = |\det(\mathbf{X} - \mathbf{X}')|^2 \quad (16)$$

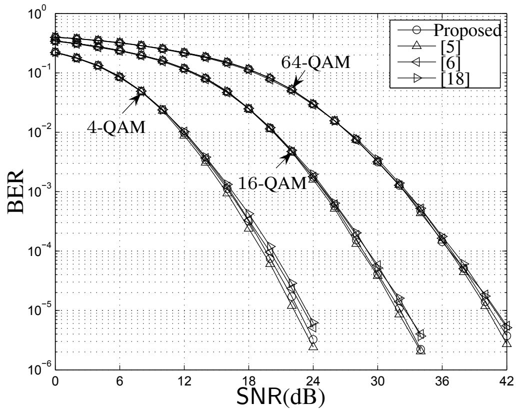

Fig. 1. Code BERs versus SNR.

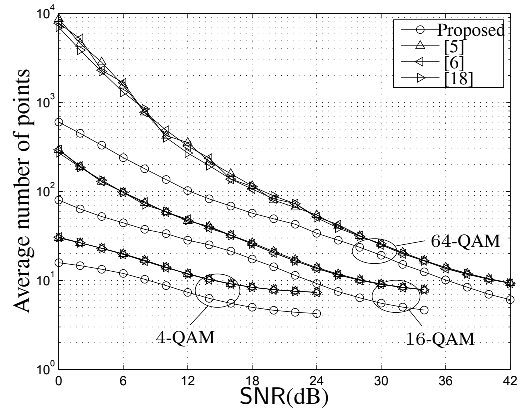

Fig. 2. Average number of points visited by the sphere decoder versus SNR.

we ensure that the designed generator matrix will provide full diversity with a high coding gain. To find the values of the angles used in the Givens rotations-based parameterization of $\tilde{\mathbb{G}}$, an extensive Monte-Carlo search was performed, followed by local optimization around the resulting from this search maximal value of the objective function. The constellation was fixed to be 4-QAM. As a result of our numerical optimization, the following matrix $\tilde{\mathbb{G}}$ has been found

$$\tilde{\mathbb{G}}_{\rm opt} = \frac{1}{\sqrt{7}} \begin{bmatrix} -1 & 1 & 1 & 2 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & 2 & -1 \\ 2 & 1 & -1 & 1 \end{bmatrix}$$

with the optimal point of the objective function $\delta(\tilde{\mathbb{G}}_{\rm opt}) = 16/7$. We stress here that the resulting $\tilde{\mathbb{G}}_{\rm opt}$ has an intriguingly simple structure that we could not expect when formulating the optimization problem in (16).

4. NON-VANISHING DETERMINANT PROPERTY

In this section, we establish the so-called non-vanishing determinant (NVD) property for our code which means that $\delta \geq c$ for any constellation size, where c is a constant. The NVD property has been established for several popular STBCs [5], [7], [8], [13]-[15]. It has been proven that any space-time code satisfying the latter property achieves the diversity-multiplexing gain tradeoff not only in the Rayleigh fading channel case [8], [13], but also for an arbitrary fading distribution [16].

The following theorem establishes the NVD property for the proposed code.

Theorem 1: For the proposed space-time code, $\delta(\tilde{\mathbb{G}}_{\rm opt}) \geq 16/7$ provided that the information symbols are drawn from any M-QAM or M-PAM constellation.

Proof: See [17]. ■

As the established non-vanishing determinant property holds true for any M-QAM constellation, our code is expected to provide a good performance for arbitrary M (even though it has been designed for the particular 4-QAM case).

5. THE DECODER

Let us consider the sphere decoder presented in [10] which is commonly used to implement the ML decoding procedure. This sphere decoder conducts a search of possible candidates of $\underline{\mathrm{s}}$ by going through the axes of $\mathbb{HG}$ in a predetermined order. We choose this order so that the axes belonging to the OSTBC (i.e., corresponding to $\mathbb{HG}_{\rm OSTBC}$) are the last ones to be searched. In this way, after going through the first K axes using the sphere decoder, our decoding algorithm needs to conduct a K-dimensional search for the remaining K axes. As these remaining axes are defined by $\mathbb{HG}_{\rm OSTBC}$ and, therefore, they are all orthogonal to each other, the search for the remaining K dimensions is equivalent to simple symbol-by-symbol decoding. Therefore, the last K stages of the sphere decoder can be omitted and replaced by the much simpler symbol-by-symbol decoder. This substantially reduces the computational cost of the overall decoding procedure as compared to the existing full-rate STBCs that use the sphere decoder to search over all $2K$ axes of the lattice.

6. SIMULATIONS

We have assumed a MIMO system with quasi-static flat Rayleigh fading channel and $N_t = N_r = T = 2$. Figure 1 shows the bit error rate (BER) versus SNR for the proposed STBC, the Golden code of [5], the code of [6], and the code of [18]. All these codes have been tested for 4-QAM, 16-QAM and 64-QAM constellations. As recommended in [18], the parameter of the latter code is chosen to be $\lambda = 1/2$ for the 4-QAM constellation, and $\lambda = \pi/6$ for the 16-QAM and 64-QAM constellations. As can be seen from Figure 1, the performance of the proposed code is nearly the same as of the Golden code (which has the best performance among the codes tested).

Figure 2 shows the average number of points visited by the sphere decoder versus SNR for the STBCs tested in Figure 1. From Figure 2, it is clear that the proposed code offers a reduced decoding complexity as compared to the other codes tested. This reduction in complexity is especially pronounced in a practically important SNR region of 0 to 15 dB.

7. CONCLUSIONS

A new 2×2 full-rate full-diversity information-lossless space-time block code has been proposed. Our code has been shown to satisfy the non-vanishing determinant property and, therefore, to achieve the diversity-multiplexing gain tradeoff. Our simulation results have demonstrated that the proposed code has a performance comparable to that of the best known full-rate STBC schemes but, at the same time, enjoys a substantially reduced decoding complexity with respect to these known schemes.

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