SPATIAL DEGREES OF FREEDOM OF CORRELATED MULTIPATH

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ABSTRACT

In this paper we consider the spatial degrees of freedom in the context of a multi-antenna wireless communication system. We investigate how the degrees of freedom depend on important system parameters such as the spatial extent of a region containing the antennas and, more importantly, the angular correlation of multipath. These results naturally augment known results which show how the degrees of freedom are affected by multipath from a restricted range of angles. We clarify the distinction between the spatial degrees of freedom with respect to an orthonormal basis and the concept of richness of multipath which is related to the Karhenen Loeve expansions for a random multipath field.

***Index Terms*—** Degrees of freedom, rich multipath, multipath modeling, correlated multipath.

1. INTRODUCTION

The degrees of freedom (DoF) of multipath is a central notion which affects the degree of spatial diversity in multi-antenna wireless communication systems [1] (and closely related problems such as direction of arrival estimation). In the analogous context of communication over a bandlimited waveform channel we can see the universal importance of this notion. In quantifying the DoF, Gallagher [2, p.361] states: *"A class of functions in which any particular function can be specified by n real numbers is said to have n degrees of freedom."* For the waveform channel with bandwidth W and transmission interval T, the degrees of freedom are fundamentally limited to $2WT$ [2, Ch.8]. The context of this paper is towards developing narrowband spatial channel analogies of the bandlimited waveform channel results: What are the spatial degrees of freedom and how do they depend on the important system parameters such as the spatial size, multipath angular diversity and multipath correlation?

If we consider a spatial signal, representing the distribution of energy in space due to sources, then any *constraint* on the signal or sources naturally reduces the degrees of freedom. If the spatial signal is generated as a consequence of signal propagation in free space then this explicitly imposes a linear constraint where the signal must satisfy the wave equation [1, 3] — for such a constrained spatial signal we use the terminology "wavefield". It has been well studied how the wavefield degrees of freedom are affected when there is an angular restriction (constraint) on the direction of arrival of signals such as when the energy arrives from a limited range of angles [1] or more general angular distributions [4, 5]. However, there are additional practically important factors that also significantly decrease the degrees of freedom. This papers explores the effects of two factors: 1) restricting the *size of the region of interest* where antennas are located, and, our primary focus, 2) *angular correlation* in the multipath.

The technical contributions of this paper are:

1. To theoretically determine the effect of the size of a circular region on the degrees of freedom. It is shown that the effect of changing the radius of the region is equivalent to filtering the angular multipath distribution with a specific non-ideal low pass filter.
2. To theoretically determine the effect of angular correlation for a given angular power spectrum on the degrees of freedom.
3. We develop a model for generating random wavefields according to a given angular correlation characteristic which permits a numerical investigation into the effects of correlation modeling.

2. DEGREES OF FREEDOM

A wavefield in a bounded region of interest in space can be accurately described with a small number of parameters. These parameters capture the notion of degrees of freedom (DoF). In this paper we focus on a circular region for our bounded region

$$\mathbb{B}_R = \{\boldsymbol{x} \colon \|\boldsymbol{x}\| \leq R\}$$

where R is the radius, $\boldsymbol{x}$ represents the 2D vector of spatial variables an $\|\cdot\|$ is the euclidean norm. This special shaped

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region admits a simpler formulation and can be regarded as a natural 2D analog of either the time interval, $[0, T]$, or the frequency interval, $[-W, W]$, used in the bandlimited waveform channel case. Alternatively, an arbitrary bounded shape can be contained within such a circular region and the DoF thereby bounded. For antenna arrays such a continuous region is better suited to bounding the performance of circular arrays than linear arrays.

If all wavefields in the region of interest $\mathbb{B}_R$ can be accurately expressible with a finite number of parameters $\{\beta_n\}_{n=1}^{N}$ multiplying some orthonormal basis functions $\Phi_n(\boldsymbol{x})$, we say the wavefield possesses *N DoF with respect to that basis*. Such a basis and coefficients can be regarded as truncation of a generalized Fourier Series representation. We shall discuss more fully the concept of degrees of freedom and how this relates to multipath richness and the optimal choice of orthonormal basis functions later in sections 4.3 and 4.4.

3. EFFECT OF REGION SIZE

We first introduce a natural set of basis functions for multipath (which are optimal only in the special case of isotropic multipath) to characterize the influence of region size on degrees of freedom.

Consider a multipath field, $F(\boldsymbol{x})$, in a circular two dimensional region of interest, $\mathbb{B}_R$ where R, the radius, is one important system parameter referred in section 1. Then, we can write,

$$F(\boldsymbol{x}) = \int_0^{2\pi} A(\varphi) e^{ik\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}(\varphi)} d\varphi, \quad \|\boldsymbol{x}\| \le R \tag{1}$$

where $A(\varphi)$ is the complex multipath scattering gain, the complex amplitude of a multipath originating from each direction φ, k is the wave number, $\widehat{\boldsymbol{y}}(\varphi)$ is a unit vector in the direction φ, and $\boldsymbol{x}\cdot\boldsymbol{y}$ denotes the vector dot product. $A(\varphi)$ implicitly represents a specific geometrical distribution of far-field scatterers.

In quantifying the degrees of freedom a Fourier expansion of $F(\boldsymbol{x})$ is better suited then (1). With an orthonormal expansion (using the natural inner product defined over the region of interest $\mathbb{B}_R$) the degrees of freedom can be determined by the significant Fourier coefficients which we write as $\beta_{n;R},\ n \in \mathbb{Z}$. So, following [6], write

$$F(\boldsymbol{x}) = \sum_{n=-\infty}^{\infty} \beta_{n;R}\, \Phi_{n;R}(\boldsymbol{x}), \quad \|\boldsymbol{x}\| \le R \tag{2}$$

where the natural orthonormal basis functions are given by

$$\Phi_{n;R}(\boldsymbol{x}) \triangleq \frac{i^n J_n(k\|\boldsymbol{x}\|) e^{in\varphi(\boldsymbol{x})}}{\sqrt{2\pi \mathcal{J}_n(R)}}, \tag{3}$$

$J_n(\cdot)$ is the integer order Bessel function of the first kind [7] and

$$\mathcal{J}_n(R) \triangleq \int_0^R \big(J_n(kr)\big)^2 r\, dr \tag{4}$$

is a normalizing term for the region of interest $\mathbb{B}_R$.

By combining the field expression (1), natural basis functions (3) and Jacobi-Anger expansion [8, p.32],

$$e^{i\boldsymbol{x}\cdot\widehat{\boldsymbol{y}}} = \sum_{n=-\infty}^{\infty} i^n J_n(k\|\boldsymbol{x}\|) e^{in(\varphi(\boldsymbol{x})-\varphi(\widehat{\boldsymbol{y}}))}, \tag{5}$$

and comparing with (2) we find

$$\beta_{n;R} = \sqrt{2\pi \mathcal{J}_n(R)}\alpha_n, \tag{6}$$

where $\alpha_n \in \mathbb{C}$ is the n^{th} Fourier series coefficient of $A(\varphi)$ defined as

$$\alpha_n = \int_0^{2\pi} A(\varphi) e^{-in\varphi} d\varphi. \tag{7}$$

We conclude our investigation of the effect on the region size on the degrees of freedom with the key results:

1. The effect of the region, parameterized by radius R, is equivalent to a filtering (convolution) of $A(\varphi)$. In the transform domain this is given by the multiplication (6). Therefore, the properties of (4) capture precisely the low pass, albeit non-ideal, action in the angular domain. Increasing R increases the effective bandwidth of the low pass action increasing the DoF.

2. The equivalent filtered multipath scattering gain $A(\varphi)$, (7), yields an effective multipath scattering gain,

$$A_R(\varphi) \triangleq \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \sqrt{\mathcal{J}_n(R)}\alpha_n e^{in\varphi} \tag{8}$$

   which is directly amenable to standard degree of freedom analysis [5]. The erosion of the Fourier coefficients with decreasing region size, R, directly decreases the degrees of freedom.

4. EFFECT OF CORRELATED MULTIPATH

4.1. Correlated Random Multipath Fields

We now consider the case of a random multipath field. Here we model the multipath scattering gain $A(\varphi)$, used in (1), as a random wavefield, defined by the second order quantity $\mathcal{E}\big\{A(\varphi_1)A^*(\varphi_2)\big\}$. From this we define the angular power spectrum (APS):

$$P(\varphi) \triangleq \mathcal{E}\{|A(\varphi)|^2\}, \tag{9}$$

which may be expanded as a Fourier series in coefficients $\gamma_n = \int_0^{2\pi} P(\varphi)e^{-in\varphi}\, d\varphi$. Note that, given the non-negativity of (9), these coefficients are constrained to lie in a positive convex cone. The APS describes the incoming multipath power as a function of direction [4]. Notice $P(\varphi)$ supplies no information regarding the phase of $A(\varphi)$.

To model *correlated* multipath we can augment the APS, (9), by introducing an angular correlation function $\rho(\varphi_1, \varphi_2)$ between two multipath directions φ_1 and φ_2:

$$\rho(\varphi_1, \varphi_2) \equiv \frac{\mathcal{E}\{A(\varphi_1)A^*(\varphi_2)\}}{\sqrt{\mathcal{E}\{|A(\varphi_1)|^2\}\mathcal{E}\{|A(\varphi_2)|^2\}}}.$$

With this we can study the impact of scatterer correlation on wavefield DoF. Previously in the literature, it has been assumed (explicitly or implicitly) that the multipaths from different directions are uncorrelated, i.e., $\rho(\varphi_1, \varphi_2)$ is non-zero if and only if $\varphi_1 = \varphi_2$ [4]. Here we consider a model for the scatterer correlation of the following form $\rho(\varphi_1, \varphi_2) = \rho(\varphi_1 - \varphi_2)$. With such angular correlation the random wavefield is constrained (relative to the uncorrelated case) and, therefore, this reduces the degrees of freedom we can expect.

To investigate the impact on DoF quantitatively we develop a model for such correlation which is amenable to numerical study. This is the subject of the next subsection.

4.2. Correlated Scatterers Model

To introduce correlated scattering, for a prescribed APS $P(\varphi)$, we propose a simple model by assuming $A(\varphi)$ is circularly complex Gaussian and generated according to

$$A(\varphi) = \sqrt{\frac{P(\varphi)}{2}}\, x(\varphi) + i\sqrt{\frac{P(\varphi)}{2}}\, y(\varphi) \qquad (10)$$

where $x(\varphi)$ and $y(\varphi)$ are independent stationary Gaussian random processes of zero mean and variance one. Note that, $P(\varphi)$ shapes the angular distribution independent of the component process correlations.

We shall work with a sampled version of (10) which simplifies introduction of the correlation. Evaluating $A(\varphi)$ at a number of angles $\varphi_1, \varphi_2, \cdots, \varphi_n$, define the vectors of correlated Gaussian random variables $\boldsymbol{x} = [x(\varphi_1), \cdots, x(\varphi_n)]^T$ and $\boldsymbol{y} = [y(\varphi_1), \cdots, y(\varphi_n)]^T$. Sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ both have the same $n \times n$ covariance matrix where each component is identified with $\rho(\varphi_i, \varphi_j)$. Generating such correlated random vectors is standard, e.g., as given in [9, p.215].

In a real multipath environment we expect that there should be correlation whenever the two angles φ_i and φ_j are close because they are likely to be coming from the same physical scatterer illuminated by the same source. Further, as the separation between the angles increases we expect decreasing correlation and so we adopt a simple model [10]

$$\rho(\varphi_i, \varphi_j) = e^{-(\varphi_i - \varphi_j)^2/2\sigma_S^2}, \quad i, j = 1, 2, \ldots, n \qquad (11)$$

where $\sigma_S \geq 0$ is a correlation spread factor, another important system parameter referred in section 1. Varying σ_S changes the amount of multipath correlation and hence the DoF. This correlation modeling is independent of the APS $P(\varphi)$.

For a uni-modal APS $P(\varphi)$ the von-Mises power distribution is the most common choice

$$P(\varphi) = \frac{1}{2\pi I_0(\kappa)} e^{\kappa \cos(\varphi - \varphi_0)}, \quad |\varphi - \varphi_0| \leq \pi, \qquad (12)$$

where κ describes the angular spread of multipath power and φ_0 is the central angle of arrival.

4.3. Evaluation of Degrees of Freedom

Previously we qualitatively linked DoF with the number of significant Fourier coefficients. Here we make the connection explicit. Consider a particular realization $F(\boldsymbol{x})$ of a random wavefield. Let $F_N(\boldsymbol{x})$ be the Nth order truncation of wavefield, (2), retaining the $2N + 1$ lowest order terms. The normalized mean square error between $(\boldsymbol{x})$ and $F_N(\boldsymbol{x})$ over $\mathbb{B}_R$ can be written

$$\epsilon_{N,R} \triangleq \frac{\int_{\mathbb{B}_R} |F(\boldsymbol{x}) - F_N(\boldsymbol{x})|^2 d\boldsymbol{x}}{\int_{\mathbb{B}_R} |F(\boldsymbol{x})|^2 d\boldsymbol{x}} = \frac{\sum_{|n|>N} |\beta_{n;R}|^2}{\sum_{n=-\infty}^{\infty} |\beta_{n;R}|^2} \qquad (13)$$

where the denominator is the total energy in the receiver region. We define the DoF as the number of parameters required for which the error between truncated and actual wavefields is below an acceptable threshold level ϵ_0:

$$\mathrm{DoF} = 2 \times \arg\min_N \{\epsilon_{N,R} \leq \epsilon_0\} + 1 \qquad (14)$$

Conventionally one takes $\epsilon_0 = 0.01$. DoF measures the number of parameters with respect to *the natural basis functions* (3). A stochastic notion of degrees of freedom can be obtained by inserting expectations around the numerator and denominator of (13) and estimated by averaging over a number of trials.

In the above formulation it is interesting to consider the question of what choice of basis leads to the least number of parameters. This question is answered in the next section and captures the notion of multipath richness.

4.4. Multipath Richness

In a random wavefield, each wavefield coefficient $\beta_{n;R}$ in (2) is a random variable. Depending upon the statistical class of wavefield, the wavefield coefficients are generally correlated (which represents redundancy). It is possible in this case to express the wavefield in a different set of basis functions with a lesser number of parameters that are uncorrelated (to the same truncation accuracy). The optimal representation of a wavefield is given by the Karhunen-Loeve (KL) expansion

$$F(\boldsymbol{x}) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \zeta_n \, \Psi_n(\boldsymbol{x}), \qquad (15)$$

(which can be truncated to the desired finite number of terms) where the orthonormal basis (eigen-)function set $\{\Psi_n\}$ represents the optimal set for a stochastic multipath field, $\lambda_n > 0$

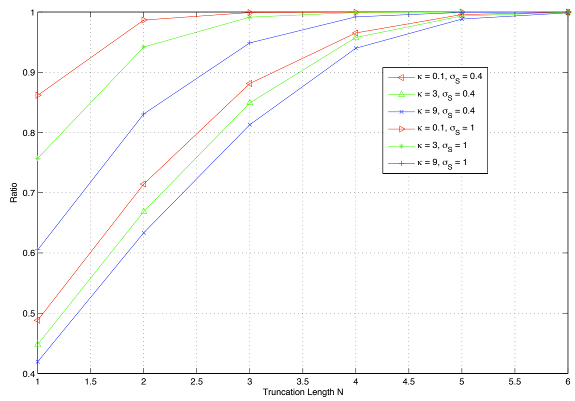

Fig. 1. Ratio $\sum_{n=-N}^{N} |\beta_n|^2 / \sum_{n=-\infty}^{\infty} |\beta_n|^2$ vs truncation order N in (13) for von-Mises APS with different angular power spread and two values for the correlation spread factor. The receiver region has radius $R = \lambda$.

represents an eigenvalue associated with eigenfunction $\Psi_n(\boldsymbol{x})$ and ζ_n are new uncorrelated wavefield coefficients (random variables) of unit variance.

We define multipath richness consistently with (13) and (14) as follows. Assume the λ_n are arranged in descending order. The field in $\mathbb{B}_R$ is said to have a *multipath richness* of N when N is the least integer such that

$$\frac{\sum_{n>N} \lambda_n}{\sum_{n=0}^{\infty} \lambda_n} < \epsilon_0. \tag{16}$$

Again conventionally one takes $\epsilon_0 = 0.01$. Under this framework, given the ℓ_2 formulation, the N in (16) tightly lower bounds the N in (13) for the same error threshold and, thereby, the richness is a tight lower bound on the degree of freedom for any basis. This framework confirms that multipath richness is proportional to the angular spread of multipath power [5]. For example, the von-Mises power distribution in (12) will generate richer multipathfor smaller κ.

4.5. Numerical Results

Fig. 1 shows the effect on the ratio (13) vs truncation order, and hence the DoF, numerically. In the lower three curves, the ratio tends to increase as κ decreases, under the same level of angular correlation of $A(\varphi)$ where $\sigma_S = 0.4$. This indicates the energy of the wavefield concentrates to low-order coefficients of $\{\beta_{n,R}\}$ sequence with richer multipath as referred in 4.4. Thus the DoF is decreased correspondingly. We further increase the angular correlation by setting σ_S equal to 1.0 in the upper three curves in Fig. 1. With a more correlated $A(\varphi)$, $\{\beta_{n,R}\}$ sequence indicates a more intense concentration to low-order Fourier coefficients $\{\beta_{n,R}\}$. This shows angular correlation contributes positively to the low-order modes energy concentration of the Fourier coefficients. This confirms the speculation that highly correlated wavefields have less DoF.

5. CONCLUSIONS

Multipath restricted to a limited range of angles is known to limit the degrees of freedom but it is further reduced if it is also correlated in angle. We have developed a model for such correlated multipath and shown how to compute the degrees of freedom in terms of the significant Fourier series coefficients of a correlated multipath wavefield expansion. We also showed that, for a circular region, the effect of changing the radius of the region is equivalent to filtering the angular multipath distribution with a specific non-ideal low pass filter.

6. REFERENCES

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