SNR ESTIMATION IN NAKAGAMI FADING CHANNELS WITH ARBITRARY CONSTELLATION

Sohail A. Dianat
Rochester Institute of Technology Rochester NY 14623
sadeee@rit.edu

ABSTRACT

In this paper, a novel technique is proposed to estimate the average signal-to-noise ratio (SNR) for Nakagami-*m* fading channels with an arbitrary signal constellation. The Nakagami-*m* distribution is a good fit to empirical fading data obtained from radio communication channels. The proposed algorithm uses absolute second and fourth moments of the envelope of the received signal over a block of data as the sufficient statistics. The estimator is blind and no training sequence is used. It is also independent of signal constellation.

Index Terms — Nakagami-m, fading, signal-to-noise, estimation

I. INTRODUCTION

The fading phenomenon in communication channels is due to the presence of time varying multipaths. The Nakagami-*m* distribution has application in the modeling of the path gain of fading channels. The parameter *m* can be varied to model channel fading conditions from no fading (nonrandom gain) to extremely severe fading [1]. An important metric of fading channels is the average SNR of the channel. The average SNR is used in applications such as Turbo decoding algorithm [2] and in maximum and optimum diversity combining algorithms that are used in multiple-input-multiple-output (MIMO) channels. In [3], an SNR estimation technique for Nakagami-*m* fading channels with BPSK modulation has been presented. In this paper, an SNR estimator for Nakagami-*m* fading channels with arbitrary constellation is proposed. The Rayleigh fading and AWGN channels are special cases corresponding to $m=1$ and $m=\infty$, and will be discussed as special cases of the proposed algorithm.

2. CHANNEL MODEL

Consider a digital data communication system over a fading channel given by

$$r_n = g_n s_n + w_n \qquad (1)$$

where r_n is the received signal at the output of the matched filter detector, s_n is the transmitted symbol of an arbitrary constellation, g_n is the channel fading gain, which is assumed to be a zero mean complex-valued process $g_n = |g_n| e^{j\phi}$,where ϕ is uniformly distributed between $-\pi$ and π , and $|g_n|$ is independent of ϕ with Nakagami distribution with parameter m The Nakagami-*m* distribution is given by [1]

$$f_{|g_n|}(|g_n|) = \frac{2}{\Gamma(m)}\left(\frac{m}{\alpha_g^2}\right)^m |g_n|^{2m-1} \exp\left(-\frac{m|g_n|^2}{\alpha_g^2}\right) \qquad (2)$$

where $\alpha_g^2 = E(|g_n|^2)$, $\Gamma(m)$ is the Gamma function and $m \ge 0.5$ is the Nakagami fading parameter. The process w_n is additive white zero mean complex Gaussian noise independent of g_n with real and imaginary parts having equal variance σ_w^2. It is assumed that the arbitrary constellation has symbols with Q different amplitudes $A_i, i=1,2,\cdots,Q$ and probabilities $p_i, i=1,2,\cdots,Q$. Here it is assumed that $Q>1$. This will exclude the M-ary PSK constellation for which $Q=1$ .The Nakagami-*m* distribution covers a variety of distributions via parameter m . For example, $m=1$ corresponds to Rayleigh distribution, $m=0.5$ indicates one sided Gaussian and as $m \to \infty$, the Nakagami-*m* fading channel approaches a static channel and the PDF of g becomes $f_{|g_n|}(x) = \delta(x-\alpha_g)$ .The average signal to noise ratio (*SNR*) corresponding to the received signal r_n is defined as

$$SNR = \frac{\alpha_g^2}{2\sigma_w^2}\sum_{i=1}^{Q} p_i A_i^2 \qquad (3)$$

The goal of this paper is to estimate the average SNR from a block of the received signal.

3. SNR ESTIMATOR

In this section, a blind SNR estimator based on absolute moments of the received signal is presented. First, we derive an expression for the k^{th} absolute moment of the received signal $|r_n|$ and use second and forth moments to form the necessary statistics to estimate the average SNR. The probability density function (PDF) of the envelope of r_n conditioned on g_n and s_n is given by [4]

$$f(|r_n| \,|\, |g_n|, s_n) = \frac{|r_n|}{\sigma_w^2}\exp\left(-\frac{|g_n|^2|s_n|^2 + |r_n|^2}{2\sigma_w^2}\right) \times I_0\left(|r_n|\frac{|g_n||s_n|}{\sigma_w^2}\right) \qquad (4)$$

where $I_0(x)$ is the modified Bessel function of the first kind and zero order. The PDF of $|r_n|$ conditioned on g_n is obtained by averaging (4) over all possible values of $|s_n|$.

$$f(|r_n||g_n) = \sum_{i=1}^{Q} p_i \frac{|r_n|}{\sigma_w^2} \exp(-\frac{|g_n|^2|A_i|^2 + |r_n|^2}{2\sigma_w^2}) \times I_0(|r_n| \frac{|g_n||A_i|}{\sigma_w^2}) \quad (5)$$

The k^{th} moment of $|r_n|$ conditioned on g_n is given by

$$E[|r_n|^k | g_n] = \int_0^\infty |r_n|^k f(|r_n| | g(n) d|r_n| \quad (6)$$

This can be reduced to [2]

$$E[|r_n|^k | g_n] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \Gamma(\frac{k}{2}+1) \exp(-\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) \times {}_1F_1(\frac{k}{2}+1:1:\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) \quad (7)$$

${}_1F_1(a;b;z)$ is the confluent hypergeometric function given by

$${}_1F_1(a;b;z) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt \quad (8)$$

Therefore the unconditional k^{th} moment of $|r_n|$ is found by averaging (7) over the PDF of $|g_n|$.

$$E[|r_n|^k] = \int_0^\infty \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \Gamma(\frac{k}{2}+1) \exp(-\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) \times {}_1F_1(\frac{k}{2}+1:1:\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) f_{g_n}(|g_n|) d|g_n| \quad (9)$$

Substituting equation (8) into (9) results in

$$E[|r_n|^k] = \int_0^\infty \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \Gamma(\frac{k}{2}+1) \exp(-\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) \times \frac{\Gamma(1)}{\Gamma(-\frac{k}{2})\Gamma(\frac{k}{2}+1)} \int_0^1 \exp(\frac{|g_n|^2|A_i|^2}{2\sigma_w^2} t) t^{\frac{k}{2}} (1-t)^{-\frac{k}{2}-1} dt \times f_{g_n}(|g_n|) d|g_n| \quad (10)$$

Substituting equation (2) into equation (10) results in

$$E[|r_n|^k] = \int_0^\infty \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \exp(-\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}) \frac{\Gamma(1)}{\Gamma(-\frac{k}{2})} \times \int_0^1 \exp(\frac{|g_n|^2|A_i|^2}{2\sigma_w^2} t) t^{\frac{k}{2}} (1-t)^{-\frac{k}{2}-1} dt \times \frac{2}{\Gamma(m)} (\frac{m}{\alpha_g^2})^m |g_n|^{2m-1} \exp(-\frac{m|g_n|^2}{\alpha_g^2}) d|g_n| \quad (11)$$

Which is further reduced to

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \frac{2m^m \Gamma(1)}{\Gamma(m)\Gamma(-\frac{k}{2})} \int_0^1 t^{\frac{k}{2}} (1-t)^{-\frac{k}{2}-1} \times \int_0^\infty \frac{|g_n|^{2m-1}}{\alpha_g^{2m}} \exp(-\frac{|g_n|^2|A_i|^2}{2\sigma_w^2}(1-t) - \frac{m|g_n|^2}{\alpha_g^2}) \, d|g_n| dt \quad (12)$$

Integrating with respect to $|g_n|$ transforms equation (12) into

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \frac{m^m \Gamma(1)}{\Gamma(-\frac{k}{2})} \int_0^1 t^{\frac{k}{2}} (1-t)^{-\frac{k}{2}-1} \times \frac{1}{[m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}(1-t)]^m} dt \quad (13)$$

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \frac{m^m}{\Gamma(-\frac{k}{2})} \times \int_0^1 \frac{t^{\frac{k}{2}} dt}{(1-t)^{\frac{k}{2}+1} [m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}(1-t)]^m} \quad (14)$$

The above integral can be written in terms of the Appell function given by

$$F_1(a:b_1,b_2,c:z_1,z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{t^{a-1} dt}{t^{a+1-c} (1-tz_1)^{b_1} (1-tz_2)^{b_2}} \quad (15)$$

Thus, equation (14) is reduced to

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \frac{m^m \Gamma(1)}{\Gamma(-\frac{k}{2})} \frac{\Gamma(-\frac{k}{2})\Gamma(1+\frac{k}{2})}{\Gamma(1)(m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2})^m} \times F_1(\frac{k}{2}+1; 0, m, 1; 0, \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}})$$

$$= \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k (\frac{m}{m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}})^m \Gamma(1+\frac{k}{2}) \times F_1(\frac{k}{2}+1; 0, m, 1; 0, \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{m + \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}) \quad (16)$$

Note that $F_1(a:b_1,b_2,c:0,x) = {}_2F_1(a,b_2,c;x)$ (17)

where ${}_2F_1(a,b_2,c;x)$ is the confluent hypergeometric function. Using (17), equation (16) is transformed into

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \left(\frac{m}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right)^m \Gamma(1+\frac{k}{2}) \times {}_2F_1\left(\frac{k}{2}+1, m, 1; \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right) \quad (18)$$

The two moments corresponding to $k=2$ and 4 are

$$E[|r_n|^2] = \sum_{i=1}^{Q} 2 p_i \sigma_w^2 \left(\frac{m}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right)^m {}_2F_1\left(2, m, 1; \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right) \quad (19)$$

and

$$E[|r_n|^4] = \sum_{i=1}^{Q} 8 p_i \sigma_w^4 \left(\frac{m}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right)^m {}_2F_1\left(3, m, 1; \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right) \quad (20)$$

Define

$$y\left(\frac{\alpha_g^2}{\sigma_w^2}\right) = \frac{2E[|r_n|^2])^2}{E[|r_n|^4]} \quad (21)$$

This results in the following

$$y\left(\frac{\alpha_g^2}{\sigma_w^2}\right) = \frac{2(E[|r_n|^2])^2}{E[|r_n|^4]} = m^m \times \frac{\left[\sum_{i=1}^{Q} p_i \left(m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right)^{-m} {}_2F_1\left(2, m, 1; \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\left(m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right)^{-1}\right)\right]^2}{\sum_{i=1}^{Q} p_i \left(m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right)^{-m} {}_2F_1\left(3, m, 1; \frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\left(m+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right)^{-1}\right)} \quad (22)$$

Now, we can estimate $\frac{\alpha_g^2}{\sigma_w^2}$ as a function of y^{-1} using interpolation and an inverse look-up table (LUT).

$$\frac{\alpha_g^2}{\sigma_w^2} = y^{-1}\left\{\frac{2(E[|r_n|^2])^2}{E[|r_n|^4]}\right\} \quad (23)$$

Plots of y^{-1} for different values of m are shown in Figure 1. In practice the second and fourth order moments are not available and must be estimated using samples of the received data. That is,

$$\hat{M}_2 = \frac{1}{N}\sum_{n=1}^{N} |r_n|^2 \quad (24) \quad \text{and} \quad \hat{M}_4 = \frac{1}{N}\sum_{n=1}^{N} |r_n|^4 \quad (25)$$

where N is the data block size. We know consider two special cases: Rayleigh and AWGN channel.

1) **Rayleigh Channel**: In this case $m=1$ and the k[th] absolute moment of r_n is given by

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \left(\frac{1}{1+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right) \Gamma(1+\frac{k}{2}) \times {}_2F_1\left(\frac{k}{2}+1, 1, 1; \frac{\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}{1+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}}\right) \quad (26)$$

Since ${}_2F_1(a,1,1;x) = (1-x)^{-a}$, the above equation simplifies to

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \Gamma(\frac{k}{2}+1)\left(1+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right)^{\frac{k}{2}} \quad (27)$$

Again, if we consider the second and fourth moments of $|r_n|$, we will have

$$M_2 = E[|r_n|^2] = \sum_{i=1}^{Q} p_i 2\sigma_w^2 \Gamma(2)\left(1+\frac{\alpha_g^2 A_i^2}{2\sigma_w^2}\right) = \sum_{i=1}^{Q} p_i 2\left(\sigma_w^2 + \frac{\alpha_g^2 A_i^2}{2}\right) = 2\sigma_w^2 + \alpha_g^2 \sum_{i=1}^{Q} p_i A_i^2 \quad (28)$$

and

$$M_4 = E[|r_n|^4] = 8\sum_{i=1}^{Q} p_i \left(\sigma_w^2 + \frac{\alpha_g^2 A_i^2}{2}\right)^2 \quad (29)$$

Equation (29) can be further simplified to

$$M_4 = 8\sum_{i=1}^{Q} p_i \left(\sigma_w^2 + \frac{\alpha_g^2 A_i^2}{2}\right)^2 = 8\sum_{i=1}^{Q} p_i \left(\sigma_w^4 + \sigma_w^2 \alpha_g^2 A_i^2 + \frac{\alpha_g^4 A_i^4}{4}\right) = 8\sigma_w^4 + 8\sigma_w^2 \alpha_g^2 \sum_{i=1}^{Q} p_i A_i^2 + 2\alpha_g^4 \sum_{i=1}^{Q} p_i A_i^4 \quad (30)$$

Now we will use equations 28 and 30 and solve for σ_g^2, and σ_w^2. The solutions are:

$$\sigma_w^2 = \frac{M_2}{2} - \frac{a}{2}\sqrt{\frac{M_4 - 2M_2^2}{2(b-a^2)}} \quad (31)$$

$$\alpha_g^2 = \sqrt{\frac{M_4 - 2M_2^2}{2(b-a^2)}} \quad (32)$$

where $a = \sum_{i=1}^{Q} p_i A_i^2$ and $b = \sum_{i=1}^{Q} p_i A_i^4$.

It is obvious that $b \geq a^2$ for any constellation. However, in the case of M-ary PSK, A_i is a constant and therefore $b = a^2$ and $M_4 = 2M_2^2$, causing the algorithm to fail. Therefore, the estimator is valid for any arbitrary constellation except M-ary PSK. The estimated SNR is then given by

$$SNR = \frac{\alpha_g^2}{2\sigma_w^2}\sum_{i=1}^{Q} p_i A_i^2 = a\frac{\sqrt{\dfrac{M_4-2M_2^2}{2(b-a^2)}}}{M_2 - a\sqrt{\dfrac{M_4-2M_2^2}{2(b-a^2)}}} \qquad (33)$$

2) **AWGN Channel**: In this case, $m \to \infty$ and the k[th] absolute moment of r_n is given by

$$E[|r_n|^k] = \sum_{i=1}^{Q} p_i 2^{\frac{k}{2}} \sigma_w^k \Gamma(\frac{k}{2}+1)\exp(-\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) \times {}_1F_1(\frac{k}{2}+1:1:\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) \qquad (34)$$

The second absolute moment of r_n is

$$M_2 = \sum_{i=1}^{Q} p_i 2\sigma_w^2 \exp(-\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2})\, {}_1F_1(2:1:\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) \qquad (35)$$

where

$${}_1F_1(2:1:\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) = \sum_{i=0}^{\infty}\frac{\Gamma(i+2)\Gamma(1)}{\Gamma(2)\Gamma(i+1)i!}(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})^i$$
$$= \sum_{i=0}^{\infty}\frac{i+1}{i!}(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})^i = (1+\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})e^{\frac{\alpha_g^2|A_I|^2}{2\sigma_W^2}} \qquad (36)$$

Thus,

$$M_2 = \sum_{i=1}^{Q} p_i 2\sigma_w^2(1+\frac{\alpha_g^2}{2\sigma_w^2}A_i^2) = 2\sigma_w^2 + \alpha_g^2 a \qquad (37)$$

Similarly the fourth absolute moment of r_n is

$$M_4 = \sum_{i=1}^{Q} 8p_i\sigma_w^4 \exp(-\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) \times {}_1F_1(3:1:\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) \qquad (38)$$

where

$${}_1F_1(3:1:\frac{\alpha_g^2|A_i|^2}{2\sigma_w^2}) = \sum_{i=0}^{\infty}\frac{\Gamma(i+3)\Gamma(1)}{\Gamma(3)\Gamma(i+1)i!}(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})^i$$
$$= \sum_{i=0}^{\infty}\frac{(i+1)(i+2)}{2i!}(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})^i \qquad (39)$$
$$= [1+2\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2}+\frac{1}{2}(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})^2]\exp(\frac{\alpha_g^2|A_i|^2}{2\sigma_W^2})$$

Therefore,

$$M_4 = \sum_{i=1}^{Q} 8p_i\sigma_w^4(1+\frac{\alpha_g^2}{\sigma_w^2}|A_i|^2 + 0.125\frac{\alpha_g^4}{\sigma_w^4}|A_i|^4) \qquad (40)$$

or

$$M_4 = 8\sigma_w^4 + 8\alpha_g^2\sigma_w^2 a + \alpha_g^4 b$$

Equations 37 and 40 are used to solve for σ_g^2 and σ_w^2. Hence,

$$\sigma_w^2 = \frac{M_2}{2} - \frac{a}{2}\sqrt{\left|\frac{M_4-2M_2^2}{b-2a^2}\right|} \quad \text{and} \quad \alpha_g^2 = \sqrt{\left|\frac{M_4-2M_2^2}{b-2a^2}\right|}$$

In practice, M_2 and M_4 are replaced by their estimates.

4. SIMULATION AND NUMERICAL RESULTS

To demonstrate the proposed algorithm, consider a Nakagami-*m* fading channel with Nakagami parameter $m = 0.6$ and 16-QAM constellation. The average SNR is changed from 4 dB to 10 dB in steps of 2. The Monte Carlo simulation results are shown in Table 1. It is observed that the estimated average SNR is close to the true SNR and the variance of the error remains small.

True SNR (dB)	4	6	8	10
Estimated SNR	3.93	5.78	8.12	10.04
MSE	0.103	0.32	0.23	0.37
Variance σ_e^2	0.1	0.43	0.56	0.57

Table 1. Simulation Results

5. CONCLUSION

In this paper, an algorithm has been developed to estimate the average SNR of a Nakagami-*m* fading channel in the presence of additive white Gaussian noise. The proposed algorithm works for any arbitrary constellation except M-ary PSK. The inputs to the algorithm are the second and fourth absolute moments of the envelope of the received signal. These statistics are estimated by calculating sample moments of the envelope of the received signal over a block of data. The algorithm assumed knowledge of the Nakagami parameter *m*. Work is underway to come up with an algorithm for joint estimation of the average SNR and the Nakagami parameter *m*.

6. References

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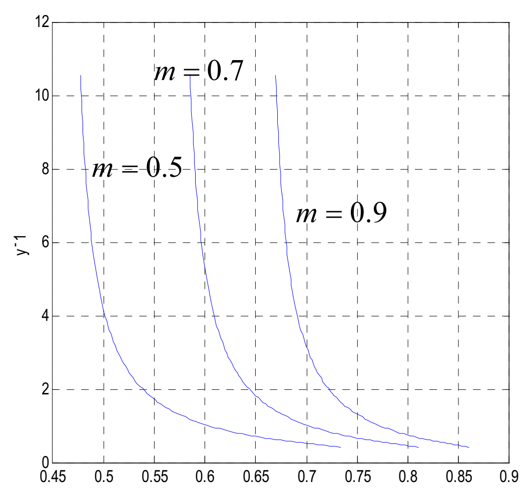

Figure 1: Plot of y^{-1}