FAST COMPUTATION OF THE HIGH RESOLUTION IMAGE RESTORATION BY USING THE DISCRETE COSINE TRANSFORM

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ABSTRACT

The high resolution image restoration method from a downsampled low resolution image by using the discrete cosine transform (DCT) is proposed. The downsampling process is modeled in matrix form and the similar transformation by the DCT matrix makes the downsampling matrix to be a sparse matrix. Then the restored high resolution image can be expressed in a scalar form by using the similar transformation and efficiently computed from the low resolution image. Computer simulations show that the proposed method is superior to the cubic spline interpolation in the high resolution image restoration performance.

Index Terms— Image restoration, Image resolution, Discrete cosine transforms

1. INTRODUCTION

High resolution (HR) image restorations, which are acquiring a HR image from downsampled low resolution (LR) images, attract many attentions [1]. In the HR image restoration methods, the downsampling process, which expresses the relation between the pixel values of the LR and HR images, is mathematically modeled. When using the charge coupled device (CCD) to acquire images, we can express the pixel value of the LR image as the sum of corresponding pixel values of the HR image [2]. Then the HR image restoration becomes an inverse problem of the downsampling process. HR image restoration method from single LR image has been derived [3] from the analogy with the blurred image restoration method [4]. In the blurred image restoration, a smoothness constraint such as the square sum of the discrete Laplacian of the restored image is imposed on the restored image to uniquely restore the blurred image [4]. Similarly, in the HR image restoration, the smoothness constraint is imposed on the restored HR image to determine it uniquely from the downsampled LR image. However, the direct solution of the HR image restoration problem requires a large computation time, because a large scale linear equation needs to be solved [3].

In this paper, we derive the fast computation method for the HR image restoration by using the discrete cosine transform (DCT). We model the downsampling process in the absence of the observation noise, perform the similar transformation of the downsampling matrix into a sparse matrix by the DCT matrix, and derive the analytical solution to the restoration problem. Since we can express the analytical solution in a scalar form, we can efficiently restore the HR image by using the DCT. We show through computer simulations that the proposed method can be computed as fast as the cubic spline interpolation method, and that the restoration capability of the proposed method is superior to that of the cubic spline interpolation method.

2. MATHEMATICAL FORMULATION OF THE IMAGE RESTORATION

2.1. Observation model

Let the $(2M \times 2M)$ high resolution (HR) image be $\{x_{ij}\}$ and the $(M \times M)$ low resolution (LR) image be $\{y_{kl}\}$. The two images are obtained from the same scene captured by the different resolution cameras at the same position. The array structures of the photodetectors of the cameras are described in Fig. 1. The density of the photodetectors of the HR image $\{x_{ij}\}$ is 4 times higher than that of the LR image $\{y_{kl}\}$. Then we can express the relation between $\{x_{ij}\}$ and $\{y_{kl}\}$ as

$$\boldsymbol{y} = H\boldsymbol{x} + \boldsymbol{v}, \tag{1}$$

where $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{v}$ are lexicographically ordered vectors consisting of the HR image $\{x_{ij}\}$, the LR image $\{y_{kl}\}$, and the additive noise with mean zero and variance ε^2, respectively. The $(M^2 \times 4M^2)$ matrix H represents the filtering and downsampling process and has the form

$$H = H_s \otimes H_s, \tag{2}$$

where "$\otimes$" denotes the Kronecker product and H_s is the $(M \times 2M)$ matrix defined by

$$H_s = \begin{pmatrix} 1/2 & 1/2 & & & & \mathbf{0} & \\ & & 1/2 & 1/2 & & & \\ & & & & \ddots & & \\ & \mathbf{0} & & & & 1/2 & 1/2 \end{pmatrix}. \tag{3}$$

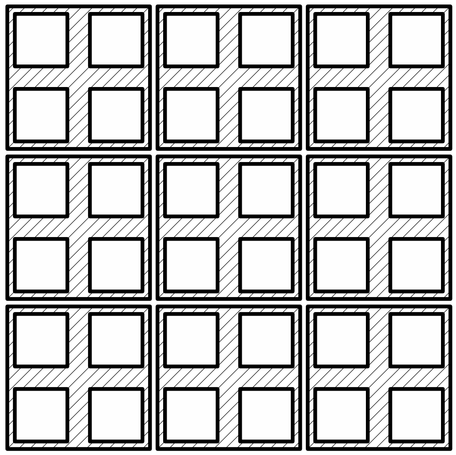

Fig. 1. Array structure of photodetectors: small squares represent the photodetectors for HR image and hatched squares represent the ones for LR image

2.2. HR image restoration problem

We shall consider the problem of restoring the HR image $\boldsymbol{x}$ from the LR image $\boldsymbol{y}$. Since the problem is ill-defined, we impose a smoothness constraint on $\boldsymbol{x}$ to uniquely determine $\boldsymbol{x}$. The constraint we imposed is the square sum of the discrete Laplacian of $\boldsymbol{x}$.

The discrete Laplacian of each x_{ij} is defined by $x_{i-1,j} + x_{i,j-1} - 4x_{ij} + x_{i,j+1} + x_{i+1,j}$. We here suppose that x_{ij}s at image boundaries $i = 0$, $i = 2M-1$, $j = 0$, or $j = 2M-1$ are under the Neuman boundary condition, that is, $x_{-1,j} = x_{0,j}$, $x_{i,-1} = x_{i,0}$, $x_{2M,j} = x_{2M-1,j}$, and $x_{i,2M} = x_{i,2M-1}$. The lexicographically ordered vector consisting of the discrete Laplacians of $\{x_{ij}\}$ is then expressed as [5]

$$P\boldsymbol{x} = (I \otimes P_s + P_s \otimes I)\boldsymbol{x}, \tag{4}$$

where I is the $(2M \times 2M)$ identity matrix, and P_s is the $(2M \times 2M)$ matrix defined by

$$P_s = \begin{pmatrix} -1 & 1 & & & \mathbf{0} \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ \mathbf{0} & & & 1 & -1 \end{pmatrix}. \tag{5}$$

Then the square sum of the discrete Laplacians of $\boldsymbol{x}$ can be represented by the quadratic form $\boldsymbol{x}^T Q \boldsymbol{x}$, where the $(4M^2 \times 4M^2)$ matrix Q is defined by

$$\begin{aligned} Q &= P^T P \\ &= I \otimes (P_s^T P_s) + 2P_s \otimes P_s + (P_s^T P_s) \otimes I. \end{aligned} \tag{6}$$

To simplify the restoration problem, we suppose that the variance of noise ε^2 is negligibly small, such as a quantization error. Then Equation (1) is approximately rewritten as

$$\boldsymbol{y} = H\boldsymbol{x}. \tag{7}$$

Under the condition (7) we shall estimate the HR image $\boldsymbol{x}$ from the LR image $\boldsymbol{y}$ by minimizing the square sum of the discrete Laplacian $\boldsymbol{x}^T Q \boldsymbol{x}$. The estimation problem is formulated as

$$\arg\min_{\boldsymbol{x}} \boldsymbol{x}^T Q \boldsymbol{x} \quad \text{subject to } \boldsymbol{y} = H\boldsymbol{x}. \tag{8}$$

We solve the problem (8) by using the Lagrange multiplier method. Let the Lagrangian L be

$$L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{x}^T Q \boldsymbol{x} + \boldsymbol{\lambda}^T(\boldsymbol{y} - H\boldsymbol{x}), \tag{9}$$

where $\boldsymbol{\lambda} = (\lambda_{00}, \lambda_{01}, \cdots, \lambda_{0,M-1}, \lambda_{10}, \cdots, \lambda_{M-1,M-1})^T$ is the M^2 dimensional vector consisting of the Lagrange multipliers. Then we have

$$\frac{\partial}{\partial \boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}) = 2Q\boldsymbol{x} - H^T \boldsymbol{\lambda} = \mathbf{0} \tag{10}$$

$$\frac{\partial}{\partial \boldsymbol{\lambda}} L(\boldsymbol{x}, \boldsymbol{\lambda}) = \boldsymbol{y} - H\boldsymbol{x} = \mathbf{0}. \tag{11}$$

Since $Q\mathbf{1} = P^T P \mathbf{1} = \mathbf{0}$, the matrix Q is found to be singular. Therefore, it is difficult to straightly solve the system of equations (10) and (11). We thus perform the similar transformations of Q and H by the discrete cosine transform (DCT) matrix, and we show that they become sparse matrices.

2.3. Similar transformations of Q and H

We put the one dimensional (1-D) DCT matrix of size M as W_M. The ij element of W_M is

$$(W_M)_{ij} = \frac{\sqrt{2}}{\sqrt{M}} c(i) \cos\left(\frac{\pi i(j+0.5)}{M}\right) \tag{12}$$

with

$$c(i) = \begin{cases} \frac{1}{\sqrt{2}} & i = 0 \\ 1 & \text{otherwise}. \end{cases} \tag{13}$$

We should note that W_M is a unitary matrix. We transform the variables $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{\lambda}$ into $\tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{y}}$, and $\tilde{\boldsymbol{\lambda}}$ by the two dimensional (2-D) DCT matrix as follows:

$$\tilde{\boldsymbol{x}} = (W_{2M} \otimes W_{2M})\boldsymbol{x} \tag{14}$$

$$\tilde{\boldsymbol{y}} = (W_M \otimes W_M)\boldsymbol{y} \tag{15}$$

$$\tilde{\boldsymbol{\lambda}} = (W_M \otimes W_M)\boldsymbol{\lambda}. \tag{16}$$

The equations (10) and (11) can be rewritten as

$$2\widetilde{Q}\tilde{\boldsymbol{x}} - \widetilde{H}^T \tilde{\boldsymbol{\lambda}} = \mathbf{0} \tag{17}$$

$$\tilde{\boldsymbol{y}} - \widetilde{H}\tilde{\boldsymbol{x}} = \mathbf{0}, \tag{18}$$

where the matrices $\widetilde{Q}$ and $\widetilde{H}$ are the similar transformations of Q and H defined by

$$\widetilde{Q} = (W_{2M} \otimes W_{2M})Q(W_{2M} \otimes W_{2M})^T \quad (19)$$
$$\widetilde{H} = (W_M \otimes W_M)H(W_{2M} \otimes W_{2M})^T. \quad (20)$$

We shall show that $\widetilde{Q}$ and $\widetilde{H}$ are sparse matrices.

We note that P_s can be diagonalized by the DCT matrix as [5]

$$\widetilde{P}_s = W_{2M} P_s W_{2M}^T = \mathrm{diag}\{\tilde{p}_0, \tilde{p}_0, \cdots, \tilde{p}_{2M-1}\}, \quad (21)$$

with

$$\tilde{p}_i = 2\left(1 - \cos\left(\frac{\pi i}{2M}\right)\right). \quad (22)$$

Using (6) and (21), we can straightforwardly show that $\widetilde{Q}$ is a diagonal matrix as follows:

$$\begin{aligned}\widetilde{Q} &= I \otimes (\widetilde{P}_s^T \widetilde{P}_s) + 2\widetilde{P}_s \otimes \widetilde{P}_s + (\widetilde{P}_s^T \widetilde{P}_s) \otimes I \\ &= \mathrm{diag}\{\tilde{q}_{00}, \tilde{q}_{01}, \cdots, \tilde{q}_{0,2M-1}, \tilde{q}_{10}, \cdots, \tilde{q}_{2M-1,2M-1}\}\end{aligned} \quad (23)$$

with

$$\tilde{q}_{ij} = (\tilde{p}_i + \tilde{p}_j)^2. \quad (24)$$

We can rewrite $\widetilde{H}$ as

$$\widetilde{H} = \widetilde{H}_s \otimes \widetilde{H}_s, \quad (25)$$

where we put

$$\widetilde{H}_s = W_M H_s W_{2M}^T = \begin{pmatrix} \tilde{h}_0 & & & & & & 0 \\ & \tilde{h}_1 & & & & & \tilde{h}_{2M-1} \\ & & \ddots & & & \cdot^{\cdot^{\cdot}} & \\ & & & \tilde{h}_{M-1} \quad 0 \quad \tilde{h}_{M+1} & & & \end{pmatrix}, \quad (26)$$

with

$$\tilde{h}_i = \begin{cases} (\frac{1}{\sqrt{2}})\cos(\frac{\pi i}{4M}) & 0 \le i \le M-1 \\ 0 & i = M, 2M \\ -(\frac{1}{\sqrt{2}})\cos(\frac{\pi i}{4M}) & M+1 \le i \le 2M-1. \end{cases} \quad (27)$$

We see from (23) and (26) that $\widetilde{Q}$ and $\widetilde{H}$ are sparse matrices.

2.4. Fast computation by the DCT

Here we show that the solution of the system of Equations (17) and (18) can be written in a scalar form. Using (26), we can rewrite Equation (18) in a scalar form as

$$\tilde{y}_{kl} = \begin{cases} \tilde{h}_0 \tilde{h}_0 \tilde{x}_{00} & k = l = 0 \\ \tilde{h}_k \tilde{h}_l \tilde{x}_{kl} + \tilde{h}_k \tilde{h}_{2M-l} \tilde{x}_{k,2M-l} & \\ \quad + \tilde{h}_{2M-k} \tilde{h}_l \tilde{x}_{2M-k,l} & \\ \quad + \tilde{h}_{2M-k} \tilde{h}_{2M-l} \tilde{x}_{2M-k,2M-l} & \text{otherwise.} \end{cases} \quad (28)$$

From the top equation in (28), we can straightforwardly determine $\tilde{x}_{00}$ as

$$\tilde{x}_{00} = \frac{\tilde{y}_{00}}{\tilde{h}_0^2}. \quad (29)$$

Therefore, we shall consider the case of $i \neq 0$ or $j \neq 0$ in the following. Substituting (23) and (25) into (17) gives

$$2\tilde{q}_{ij}\tilde{x}_{ij} = \begin{cases} 0 & i = M \text{ or } j = M \\ \tilde{h}_i \tilde{h}_j \tilde{\lambda}_{\kappa(i)\kappa(j)} & \text{otherwise,} \end{cases} \quad (30)$$

where we put

$$\kappa(i) = \begin{cases} i & 0 \le i \le M \\ 2M - i & M+1 \le i \le 2M-1. \end{cases} \quad (31)$$

Since $\tilde{q}_{ij} \neq 0$, Equation (30) is rewritten as

$$\tilde{x}_{ij} = \begin{cases} 0 & i = M \text{ or } j = M \\ \frac{\tilde{h}_i \tilde{h}_j \tilde{\lambda}_{\kappa(i)\kappa(j)}}{2\tilde{q}_{ij}} & \text{otherwise.} \end{cases} \quad (32)$$

Putting (32) into the bottom equation of (28), we have

$$\tilde{\lambda}_{kl} = \frac{2\tilde{y}_{kl}}{D_{kl}}, \quad (33)$$

where we put

$$\begin{aligned} D_{kl} = {} & \tilde{h}_k^2 \tilde{h}_l^2 \tilde{q}_{kl}^{-1} + \tilde{h}_k^2 \tilde{h}_{2M-l}^2 \tilde{q}_{k,2M-l}^{-1} + \tilde{h}_{2M-k}^2 \tilde{h}_l^2 \tilde{q}_{2M-k,l}^{-1} \\ & + \tilde{h}_{2M-k}^2 \tilde{h}_{2M-l}^2 \tilde{q}_{2M-k,2M-l}^{-1}. \end{aligned} \quad (34)$$

Here we used the property that $D_{\kappa(i),\kappa(j)} = D_{ij}$ for $0 \le i, j \le 2M-1$. Substituting (33) into (30), and summarizing it together with the result for the case $i = j = 0$ given by Equation (29), we have

$$\tilde{x}_{ij} = \begin{cases} \dfrac{\tilde{y}_{00}}{\tilde{h}_0 \tilde{h}_0} & i = j = 0 \\ 0 & i = M \text{ or } j = M \\ \dfrac{\tilde{h}_i \tilde{h}_j}{\tilde{q}_{ij} D_{ij}} \tilde{y}_{\kappa(i)\kappa(j)} & \text{otherwise.} \end{cases} \quad (35)$$

This equation shows that the analytical solution of the HR image restoration problem can be written in a scalar form.

Now we describe the algorithm of the proposed method in Fig. 2. We can compute $\tilde{\boldsymbol{x}}$ from $\tilde{\boldsymbol{y}}$ by using (35) in $O(M^2)$. The DCT of $\boldsymbol{y}$ and the inverse DCT of $\tilde{\boldsymbol{x}}$ can be computed in $O(M^2 \log M)$. Therefore, the proposed method can compute the HR image $\boldsymbol{x}$ from the LR image $\boldsymbol{y}$ in $O(M^2 \log M)$ processing time.

3. SIMULATION RESULTS

In this section, we compared the restoration performances of the cubic spline interpolation and proposed methods. All simulations were done on an IBM PC/AT compatible computer

1. Compute the DCT of $\boldsymbol{y}$ by Equation (15) to obtain $\tilde{\boldsymbol{y}}$
2. Compute $\tilde{\boldsymbol{x}}$ from $\tilde{\boldsymbol{y}}$ by Equation (35)
3. Compute the inverse DCT of $\tilde{\boldsymbol{x}}$ by Equation (14) to obtain $\boldsymbol{x}$

Fig. 2. Algorithm of the proposed method

with an Intel Pentium 4 2.4 GHz and 512 Mbyte DRAM's. We used eight images of size (256×256) with 8 bit grayscale. We generated the (128×128) LR image $\boldsymbol{y}$ from the original (256×256) HR image $\boldsymbol{x}$ by using Equation (7). Then we restored the (256×256) HR image from $\boldsymbol{y}$ by the cubic spline interpolation and the proposed methods. Fig. 3 shows the restored images of "barbara". The restored image by the proposed method seems to be more "high-passed" than the other. We then quantitatively measured the restoration performance of each method by the peak signal to noise ratio (PSNR) defined by

$$\mathrm{PSNR} = 10 \log_{10} \left(\frac{255^2}{\frac{1}{4M^2} \sum_{i=0}^{2M-1} \sum_{j=0}^{2M-1} e_{ij}^2} \right), \qquad (36)$$

where e_{ij} is the difference of the pixel value between the original and restored images. The PSNRs of the proposed method were superior to that of the cubic spline interpolation method in all the eight images we tested. The average PSNRs of the proposed and the cubic spline interpolation method were 28.36 dB and 28.08 dB, respectively. The computation times of the proposed and cubic spline interpolation methods were 0.011 sec and 0.0052 sec, respectively. The proposed method achieves better restoration performance at the expense of an increase of computation time.

(a)

(b)

Fig. 3. Restored images of "barbara" by using (a) the proposed method (PSNR = 28.06dB) and (b) the cubic spline interpolation method (PSNR = 27.86dB)

4. CONCLUSION

We derived the HR image restoration method from the downsampled LR image by using the DCT. The restoration performance of the proposed method is superior to that of the cubic spline interpolation at the expense of an increase of computation time.

5. REFERENCES

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