# NONLINEAR BISTABLE STOCHASTIC RESONANCE FILTERS FOR IMAGE PROCESSING *

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## ABSTRACT

Nonlinear bistable double-well stochastic resonance systems have been successfully used for one-dimensional signal processing, based on the concept of parameter-tuning stochastic resonance. This paper will investigate the applications of parameter-tuning stochastic resonance in image processing. First, a two-dimensional stochastic resonance system is introduced as a nonlinear filter for image processing. The equation satisfied by the dynamic probability density function of the images processed by this stochastic resonance filter and its solutions are then discussed. Finally, this nonlinear filter is used to process a black-white image corrupted by additive white Gaussian noise to reveal the possibility to extend the concept of parameter-tuning stochastic resonance to two-dimensional cases. This provides an innovative approach for image processing.

***Index Terms—*** Image Filtering, Stochastic Systems

## 1. INTRODUCTION

Image processing has been widely applied to the improvement on image quality. In order to enhance the images corrupted by noise, most of the denosing algorithms will try to remove the noise from the images. Stochastic resonance, on the contrary, is a phenomenon that the noise can be used to enhance the system performance. The concept of stochastic resonance was first proposed by Benzi in 1981 [1]. It has wide-range application areas, such as in physics, chemistry, biomedical sciences, and engineering systems [2]. Balance control [3] and speech understanding [4] are two of its applications. Signal detection [5], signal transmission [6], and signal estimation [7] are its applications in signal processing. The noise can become beneficial to the systems, only when the synchronization between the input signal and the noise occurs. This can be realized by either the traditional method (adding noise) [2], or by tuning system parameters without adding noise (parameter-tuning stochastic resonance) proposed by us [8]-[12][14][15]. Parameter-tuning stochastic resonance (PSR) is shown to be a better method in some cases [8]. We have already applied PSR into one-dimensional signal processing, such as recovering the noisy multi-frequency signals [9], and reducing the bit-error rate (BER) of the transmission of baseband binary signals [12]. Image processing is another potential application of stochastic resonance. There is some very initial research work on this topic based on the traditional stochastic resonance (adding noise). For some image processing tasks, it is impossible to add additional noise into the systems. This paper will investigate the applications of PSR in image processing. Based on the stochastic characteristics described by the dynamic probability density function of the images processed by the two-dimensional nonlinear stochastic resonance filter, this paper will use parameter-tuning stochastic resonance technique to process black-white images corrupted by white Gaussian noise to reveal the possibility to extend the concept of parameter-tuning stochastic resonance to two-dimensional cases. This provides an innovation approach for image processing.

The rest of this paper is organized as follows. In Section 2, a nonlinear two-dimensional stochastic resonance system and the equation satisfied by the dynamic probability density function of the images processed by this system are introduced. Its stationary and expanding solutions are discussed in Section 3. A black-white image corrupted by additive white Gaussian noise is processed by this stochastic resonance filter in Section 4. Finally, Section 5 closes the paper with brief concluding remarks and future research directions.

## 2. TWO-DIMENSIONAL STOCHASTIC RESONANCE SYSTEM AND EQUATION OF DYNAMIC PROBABILITY DENSITY FUNCTION

The one-dimensional nonlinear bistable stochastic resonance system can be described by the following equation [2]

$$\dot{x}(t) = ax(t) - bx^3(t) + s(t) + \eta(t), \tag{1}$$

where $a$ and $b$ are system parameters, $s(t)$ is an input signal, and $\eta(t)$ is additive noise.

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Similarly, we can propose the following two-dimensional nonlinear bistable dynamic system

$$\frac{\partial^2 w}{\partial x \partial y} = -\gamma \frac{\partial w}{\partial x} + f(w) + \Gamma(x, y), \quad (2)$$

where $w = w(x, y)$ is the state variable (system output), $f(w) = aw - bw^3 + h$, $\Gamma(x, y)$ is additive white Gaussian noise, $h$ is an input signal, and $\gamma$ is a positive damping coefficient.

We can reduce the partial differential equation (2) to an ordinary differential equation along the line $x = x_0 + t\Delta x, y = y_0 + t\Delta y$ as

$$\frac{d^2 w}{dt^2} = -\gamma \Delta y \frac{dw}{dt} + \Delta x \Delta y f(w) + \Delta x \Delta y \Gamma(x_0 + t\Delta x, y_0 + t\Delta y), \quad (3)$$

where

$$\langle \Gamma(x, y)\Gamma(x_1, y_1)\rangle = 2D\delta(x - x_1, y - y_1). \quad (4)$$

The equation (3) can then be rewritten as

$$\frac{dw}{dt} = \Delta x v, \quad \frac{dv}{dt} = -\gamma \Delta y v + \Delta y f(w) + \frac{1}{\Delta x}\Gamma_1(t), \quad (5)$$

where

$$\langle \Gamma_1(t)\Gamma_1(t_1)\rangle = 2D\delta(t - t_1). \quad (6)$$

Based on the concept in [13], we can prove the probability density function $\rho(w, v, t)$ satisfies the following equation which is called Fokker-Planck equation (FPE) for one-dimensional cases:

$$\frac{\partial \rho(w, v, t)}{\partial t} = -\frac{\partial}{\partial w}[\Delta x v \rho(w, v, t)] - \frac{\partial}{\partial v}[\Delta y(-\gamma v + f(w))\rho(w, v, t)] + D\frac{1}{\Delta x}\frac{\partial^2 \rho(w, v, t)}{\partial v^2}. \quad (7)$$

## 3. STATIONARY AND EXPANDING SOLUTIONS OF TWO-DIMENSIONAL FPE

Let $\rho_0(w, v)$ be the stationary solution of (7). It will satisfy the following equation

$$-\frac{\partial}{\partial w}[\Delta x v \rho_0(w, v)] - \frac{\partial}{\partial v}[\Delta y(-\gamma v + f(w))\rho_0(w, v)] + D\frac{1}{\Delta x^2}\frac{\partial^2 \rho_0(w, v)}{\partial v^2} = 0. \quad (8)$$

Assume $\rho_0(w, v) = e^{-a_0 v^2}\varphi(w)$, we can obtain

$$\rho_0(w, v) = e^{-\phi(w, v)}, \quad (9)$$

where

$$\phi(w, v) = \frac{\Delta x^2 \Delta y \gamma}{2D} v^2 - \frac{\Delta x \Delta y^2 \gamma}{D}\int_0^w f(w)dw - \ln N_0, \quad (10)$$

and $N_0$ is a constant for normalization.

Similar to the one-dimensional parameter-tuning stochastic resonance, the derivation of the expanding solution of (7) will depend on the calculation of the system response speed.

Assume $\rho(w, v, t) = \xi(w, v)e^{-\lambda t}$. Similar to the one-dimensional case, let

$$\xi(w, v) = \psi(w, v)e^{-\phi/2}, \quad (11)$$

where $\phi$ is defined in (10) and $e^{-\phi}$ is the stationary solution of (7).

Define the differential operator $L$ as

$$L(\rho) = -\frac{\partial}{\partial w}(v\Delta x \rho) - \frac{\partial}{\partial v}(\Delta y[-\gamma v + f(w)]\rho) + \frac{D}{\Delta x}\frac{\partial^2 \rho}{\partial v^2}. \quad (12)$$

In this case, equation (7) becomes

$$\lambda \psi = -L\psi. \quad (13)$$

Let

$$D_1 = v\Delta x, \quad D_2 = \Delta y[-\gamma v + f(w)], \quad D_{22} = \frac{D}{\Delta x}. \quad (14)$$

We denote the conjugate operator of $L$ as $L^c$. Let $L_s = (L + L^c)/2$, and $L_{as} = (L - L^c)/2$, that is $L = L_s + L_{as}$, we can derive:

$$L^c = e^{-\phi/2}[D_1 \frac{\partial}{\partial w}(e^{\phi/2}) + D_2 \frac{\partial}{\partial v}(e^{\phi/2}) + D_{22}\frac{\partial^2}{\partial v^2}(e^{\phi/2})]. \quad (15)$$

From this, we obtain

$$L_s = e^{\phi/2}\frac{\partial}{\partial v}[D_{22}e^{-\phi}\frac{\partial}{\partial v}(e^{\phi/2})], \quad (16)$$

$$L_{as} = -e^{\phi/2}[\frac{\partial}{\partial w}(D_1 e^{-\phi/2}) + \frac{\partial}{\partial v}(D_2 e^{-\phi/2}) + \frac{\partial}{\partial v}(D_{22}\frac{\partial \phi}{\partial v}e^{-\phi/2})]. \quad (17)$$

It is easy to prove the operator $L_{as}$ is anti-symmetric and $L_s$ is both symmetric and semi-definite. This means that the differential operator $L$ can be decomposed into the anti-symmetric part $L_{as}$ and the symmetric part $L_s$.

Now, we assume that $\lambda$ is any non-zero eigenvalue of the operator $-L$, and $\lambda_1^s$ is the least non-zero eigenvalue of the operator $-L_s$. It can be proven that $\lambda_1^s$ can be regarded as a lower bound of the system response speed of (2), that is

$$Re\lambda \geq \lambda_1^s, \quad (18)$$

where $\lambda_1^s$ can be calculated in an appropriate algorithm.

Based on these, we can get the expanding solution of (7)

$$\rho(w, v, t) = \rho_0(w, v) + C_1\psi_1(w, v)e^{-\lambda_1 t - \phi/2} + C_2\psi_2(w, v)e^{-\lambda_2 t - \phi/2} + ..., \quad (19)$$

where $\rho_0$ is the stationary solution described by (9), and the pairs $\{\lambda_i, \psi_i\}$, for $i = 1, 2, ...$, are the eigenvalues and eigenfunction of $-L$. It is obvious that $\rho(w, v, t_1) \approx \rho_0(w, v)$, when $\lambda_1^s t_1 \gg 1$. This can be satisfied, if the system parameters $a$, $b$, and $\gamma$ which are defined in (2) are tuned properly.

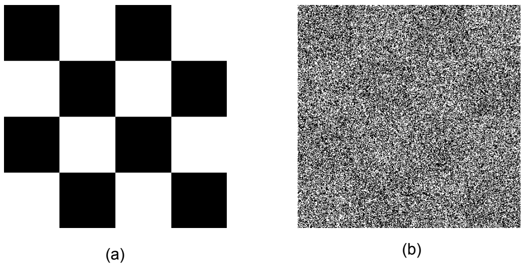


**Fig. 1**. (a) Original Black-white Image (b) Noisy Image

## 4. IMAGE PROCESSING USING PARAMETER-TUNING STOCHASTIC RESONANCE

The two-dimensional bistable system (2) can be used as a nonlinear filter to process the noisy images and improve the image quality. The stochastic characteristics of the images processed by this filter are described by (7), (9), and (19). It can then be used to derive other performance measures based on the different requirements of image processing tasks, such as image signal-to-noise ratio, probability of target detection error, etc. The performance measures can then be taken as the objective function and will be optimized when the system parameters $a$, $b$, and $\gamma$ are tuned properly to synchronize the input signal and noise and realize the stochastic resonance effect.

Now, we will apply the two-dimensional stochastic resonance system (2) to the black-white image processing which are corrupted by the additive white Gaussian noise. The black-white image $s(i,j)$ is composed of black blocks and white blocks. Each block is of length $T_b$. Also, the black pixel is represented by value $-1$ and the white pixel is represented by value 1. The image $s(i,j)$ can then be described as

$$s(i,j) = -1 \quad \text{or} \quad s(i,j) = 1,$$
$$for \quad (n-1)T_b \le i < nT_b, \quad (m-1)T_b \le j < mT_b. \quad (20)$$

We assume image $s(i,j)$ is corrupted by additive white Gaussian noise, that is

$$img(i,j) = s(i,j) + \Gamma(i,j), \quad (21)$$

where $0 \le i \le N-1, 0 \le j \le M-1$, and $\Gamma(i,j)$ is two-dimensional white Gaussian noise.

Figure 1 shows the original black-white image and the image corrupted by certain additive white Gaussian noise. From these images, we notice that the useful image signals are totally buried in the background noise and the image patterns cannot be identified directly.

The recovery of the image corrupted by noise is in fact a pattern recognition problem. It can be converted to the following detection problem

$$H_0 : img(i,j) = A + \Gamma(i,j),$$
$$H_1 : img(i,j) = -A + \Gamma(i,j), \quad (22)$$

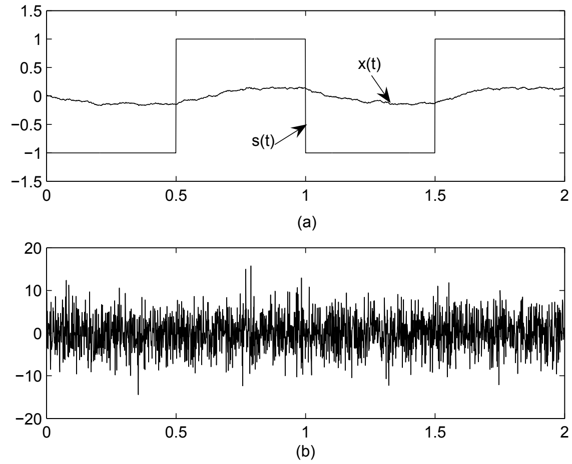


**Fig. 2**. (a) Original PAM Signal $s(t)$ and Filtered Signal $x(t)$ (b) Noisy Signal

where A=1.

If hypothesis $H_0$ is decided, the image block is then identified as white. Otherwise, the image block is taken as black.

For this detection task, we will first process the noisy black-white image with the two-dimensional stochastic resonance system (2). Then, we will developed the test statistic $T$ based on the filtered image pixel values, rather than based on the noisy image pixel values $img(i,j)$ directly.

For the one-dimensional binary PAM signals corrupted by white Gaussian noise, the related one-dimensional parameter-tuning stochastic resonance filter can be used to identify the original signal bit value, based on the filtered signal value at the point $t = T_b$, where $T_b$ is the bit duration. If the value is positive, then the original bit value is 1, otherwise the original bit value is -1. The reason is that the filtered signal value will approach the input value at the end of the bit duration, if the filter parameters are tuned properly. This is shown in Figure 2.

For the black-white images corrupted by noise, we can process them in a similar way as one-dimensional case. From Section 2, we know that the partial differential equation (2) is reduced to an ordinary differential equation along the line $x = x_0 + t\Delta x, y = y_0 + t\Delta y$. For each image block, we can make decision based on the pixel values around the bottom right corner of this block. If the average pixel value is less than zero, the image block is then taken as black. Otherwise, it is white block. This is demonstrated in Figure 3. This figure also shows the filtered images for different values of system parameters $a$, $b$, and $\gamma$. It is obvious that the filtering effect is greatly affected by the choices of system parameter values. They should be tuned optimally to get the best filtered image quality. The objective function can be the probability of error $P_e$

$$P_e = P_A P(-A|A) + P_{-A} P(A|-A), \quad (23)$$

where $P_A$ is the probability of being white block, $P_{-A}$ is the probability of being black block, $P(-A|A)$ and $P(A|-A)$

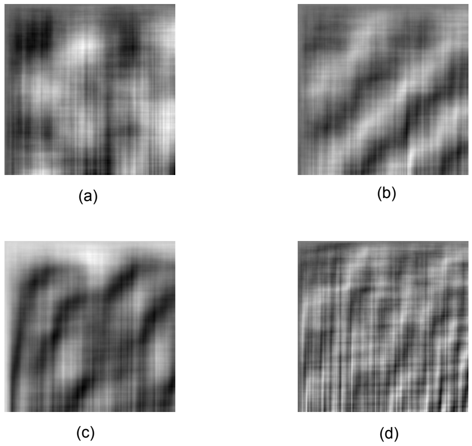


**Fig. 3**. Filtered Images (a) $a = 0.15, b = 20, \gamma = 1$ (b) $a = 0.15, b = 200, \gamma = 1$ (c) $a = 10, b = 20, \gamma = 1$ (d) $a = 0.15, b = 2000, \gamma = 1$

are the conditional probability of error detection.

$P_e$ can be expressed as a function of $\rho(w, v, t)$ which is, in turn, the function of parameters of system (2). So, we can construct the following optimization problem to optimize the system parameters and realize the stochastic resonance effect.

$$\begin{aligned} & min\ P_e, \\ \text{subject to:}\quad & a > 0, b > 0, \gamma > 0. \end{aligned} \tag{24}$$

## 5. CONCLUSION AND FUTURE WORK

This paper reveals that the one-dimensional parameter-tuning stochastic resonance can be extended to the two-dimensional case. The system parameters of the nonlinear stochastic resonance filters can be tuned optimally to realize the stochastic resonance effect and convert the noise into a positive factor to improve the image quality. The parameter-tuning stochastic resonance provides an innovative and promising approach for image processing. Next, we will implement the optimization algorithm to search for optimal system parameter values. We will also explore more applications of this new approach in image processing.

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