STABILIZED ANISOTROPIC DIFFUSIONS

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ABSTRACT

Anisotropic diffusion is an iterative process which provides efficient signal smoothing with feature preserving capabilities. However, the traditional anisotropic diffusion algorithms are highly sensitive to the number of iterations. In this paper, we introduce a novel method in the diffusion formulations to stabilize the diffusion results. It is generally applicable to most of the anisotropic diffusion algorithms, and the experimental results show that the stabilized algorithms provide improved results.

***Index Terms*—** Anisotropic diffusion, total variation, signal restoration, image enhancement, partial differential equation

1. INTRODUCTION

Filtering or smoothing is one of the key steps in signal and image processing. Many types of images have approximately piece-wise constant gray levels, which characterizes the coherence and homogeneity of an object. Anisotropic diffusion introduced in [1] is an adaptive smoothing technique which overcomes the drawbacks of the traditional linear filters, including blurring the edges, shifting the edges from its actual locations, and destroying edge junctions. It can also be used for image enhancement, segmentation, and scale-space creation. Anisotropic diffusion methods smooth images by solving a partial differential equation where the diffusion coefficient is a non-negative and non-increasing function of the magnitude of local image gradient. This formulation provides a nice adaptive filtering effect which the edges of the objects can be preserved.

Anisotropic diffusion algorithms are iterative processes for minimizing an energy functional in order to optimize the smoothing effect. However, the stopping criteria for the smoothing process is not defined and the resultant signals or images are normally flatten after a large number of iterations in traditional anisotropic diffusion algorithms [1, 2, 3, 4, 5, 6, 7]. Therefore, the optimal stopping time [8, 9, 10] for the anisotropic diffusion process becomes a known issue of the existing methods.

In this paper, we introduce a novel method in the diffusion formulations to stabilize the smoothing result. The output of the diffusion process is then converged and is insensitive to the number of iterations. This makes the definition of the stopping criteria easy, and hence, solves the termination problem.

This paper is organized as follows: In Section 2, the basic formulations of anisotropic diffusion is summarized. In Section 3, we introduce the novel stabilization method for anisotropic diffusions and define its formulations with a couple of existing diffusion algorithms in details. In Section 4, the stabilized algorithms are compared with their original algorithms on various examples to demonstrate the effectiveness of the proposed method. Finally, the conclusions are given in Section 5.

2. ANISOTROPIC DIFFUSION

Anisotropic diffusion removes noise from an image by iterative modification of the image via a partial differential equation. Perona and Malik [1] introduced the idea of nonlinear diffusion that is preferred within a smooth region to diffusion near an edge. For a given image u, the partial differential equation which they had proposed is as follows:

$$\frac{\partial u}{\partial t} = div\left[g\left(\|\nabla u\|\right)\nabla u\right] \tag{1}$$

where g is the diffusion coefficient and is a non-negative, non-increasing function of the magnitude of local image gradient. One of the typical choices of the diffusion coefficient is in (2).

$$g(\nabla u) = \exp\left[-\left(\frac{\|\nabla u\|}{k}\right)^2\right] \tag{2}$$

Since the diffusion coefficients make the diffusion process perform selective smoothing, which depends on the magnitude of the image gradient at the signal sample, the edges remain sharp and undistorted. Thus, it yields stable edges across many scales, and it is not necessary to track edges across the scale space, which is a complicated and expensive task.

Inspired by the introduction of anisotropic diffusion, there have been exploratory efforts in connecting adaptive smoothing with systems of nonlinear partial differential equations [2, 3, 4, 5, 6, 7], and several modifications of anisotropic diffusion models were made. Catté introduced a Gaussian smoothed image as the variable of the diffusion coefficient $g(|\nabla(G_\sigma * u)|)$ in [2]. Further extended their work, Alvarez modified the diffusion operator in [3] to diffuse the smoothing image u in the direction orthogonal to its gradient and does not diffuse at all in the direction of the gradient. Luo use the coupled partial differential equations in [11] to smooth regions and diffuse the image where the gradient is small, but preserve them well where the gradient is large.

However, all these anisotropic diffusion algorithms are highly sensitive to the number of iterations. The quality of the resultant signal will degrade when it is become flatten or blurred along the iterative process. As a defining characteristic, iterative operations are inevitably involved in adaptive smoothing. Thus the performance of an iterative algorithm highly depends upon the termination time, which coupled with the fact that adaptive smoothing algorithms generally converge to a uniform intensity image [1, 2, 3, 4, 5, 7], causes what we often refer to as the termination problem [12]. In other words, when and where to stop smoothing is a challenging problem and there is no explicit stopping criterion has been found yet. The termination problem indeed becomes an obstacle for the use of adaptive smoothing in practice [10].

3. STABILIZED ANISOTROPIC DIFFUSION

In order to reduce the sensitivity of the anisotropic diffusion process on the number of iterations and avoid the rapid degradation of its resultant signal quality, we introduce a stabilization method in the diffusion formulations. The idea of this stabilization method is inspired by the total variation (TV) method introduced by Rudin et al. in [13]. The TV method smooths the original signal u_o and obtains the resultant image u by minimizing the following energy functional.

$$E_{TV} = \int_\Omega |\nabla u| dxdy + \frac{\lambda_1}{2} \int_\Omega |u - u_o|^2 dxdy \qquad (3)$$

where $\Omega \subset \mathbf{R}^2$ is the domain in which the image is defined.

The first term is in fact the total variation of image u. This term is generally called a regularization term. It is used to penalize oscillations and contributes as the major smoothing force in the entire model. The second term is the fidelity term which avoids the resultant image to deviate from the original signal too far, and therefore, preserves the major features of signal.

We find that the fidelity term is robust in controlling the iterative process not only in the TV method, but also applicable in the diffusion processes of anisotropic diffusions. This term minimizes the differences between the smoothing signal and the original signal. Therefore, the smoothed results will not be flatten since it is governed by the original signal from time to time during the iterative process through this term.

We incorporate the fidelity term to a couple of anisotropic diffusion algorithms for performance evaluations. The two anisotropic diffusion algorithms are the classical reference algorithm (P&M) in [1] and the recently developed algorithm based on gradient vector flow (GVF) in [7].

The energy function of the stabilized P&M algorithm are formulated in (4).

$$E_{PM} = \int_\Omega g(|\nabla u|)|\nabla u| dxdy + \frac{1}{2} \int_\Omega |u - u_o| dxdy \qquad (4)$$

where g is defined in (2), u_o is the original noisy image, and u is the smoothing image. The Euler-Lagrange of the equation is computed and can be solved by using a dynamic scheme below:

$$\frac{\partial u}{\partial t} = div\,[g(|\nabla u|)|\nabla u|] + \frac{1}{2}|u_o - u| \qquad (5)$$

Similarly, the partial differential equations of the stabilized GVF-based algorithms are formulated in (6).

$$\frac{\partial u}{\partial t} = -\vec{v}|\nabla u| + \frac{1}{2}|u_o - u| \qquad (6)$$

where $\vec{v}$ is the gradient vector flow [7]. These stabilized anisotropic diffusion algorithms can prevent the image from getting blurred to a uniform intensity image. Hence, this can maintain the quality of the resultant image.

4. EXPERIMENTAL RESULTS

In this section, we describe our comparison methodology and present the comparative smoothing results between the original algorithms and their stabilized versions on various two-dimensional images.

4.1. Methodology

Signal-to-noise ratio (SNR) is used in evaluating performance of a smoothing algorithm. We define SNR_t be the SNR of the smoothed image at Iteration t below:

$$SNR_t = 10 \log_{10} \left\{ \frac{\sum_{(x,y)\in I} I(x,y)^2}{\sum_{(x,y)\in I} [I(x,y) - u_t(x,y)]^2} \right\} \qquad (7)$$

For a given image I, $u_o = I + \eta$ is the noisy image at Iteration 0 with noise η. u_t denotes the smoothed image at Iteration t while an adaptive smoothing algorithm is applied to u_o.

We compare two algorithms, including the anisotropic diffusion (P&M) algorithm in [1] and a recently developed gradient vector flow (GVF) based algorithm in [7], with their stabilized versions. In the comparisons, not only SNR will be used as quantitative comparisons, but also illustrations will be shown for visual quality comparisons.

4.2. Comparisons

We apply the algorithms on some two-dimensional images. Fig. 1(a) shows a real clinical magnetic resonance (MR) image. Zero mean white Gaussian noise is added as shown in Fig. 1(b) with initial SNR equals to 12dB. The algorithms are applied for 100 iterations. Fig. 1(c) and (d) show the results of the original and stabilized P&M algorithms respectively. Fig. 1(e) and (f) show the results of the original and stabilized GVF-based algorithms respectively.

It is clearly shown that the visual quality of the image can be maintained at certain levels by using the stabilized algorithms after a large number of iterations. By checking the SNR_t along the iteration process in Fig. 1(g), we find that the SNR of the original P&M algorithm decreases as it blurs the image gradually and the original GVF algorithm produces an output image which is heavily blurred. The SNRs of the results of P&M, stabilized P&M, GVF, and stabilized GVF algorithms are 13.31dB, 14.76dB, 11.04dB, and 16.45dB respectively. It is noted that the stabilized algorithms provide higher SNR than the original algorithms, especially after larger number of iterations.

Fig. 2(a) shows a noise-free benchmark image "Mandrill". Its noisy version is as shown in Fig. 2(b) with initial SNR equals to 12dB. The results of P&M, stabilized P&M, GVF, and stabilized GVF, are shown in 2(c), (d), (e) and (f) respectively. The results of original P&M is quite noisy as it spreads out the noise after 100 iterations. The stabilized P&M keeps the visual quality of the image. On the other hand, the original GVF algorithm blurs the image as in the previous examples. However, its stabilized version retains a very good level of visual quality after iterations.

In terms of SNR, the stabilized algorithms consistently provide stably higher SNR against their original algorithms. The SNRs of P&M, stabilized P&M, GVF, and stabilized GVF algorithms are 12.67dB, 13.70dB, 13.53dB, and 16.03dB respectively.

5. CONCLUSIONS

Anisotropic diffusion is an efficient technique in image processing for removing noise while preserving image features. Using our proposed stabilization method, the originally iteration sensitive diffusion process becomes iteration insensitive. This method is applicable to various anisotropic diffusion algorithms for performance improvements. The stabilized anisotropic diffusion algorithms not only provides higher SNR, but also provides better visual quality in image smoothing. Moreover, their iteration insensitive characteristic improve the feasibility of the anisotropic diffusion algorithms.

6. REFERENCES

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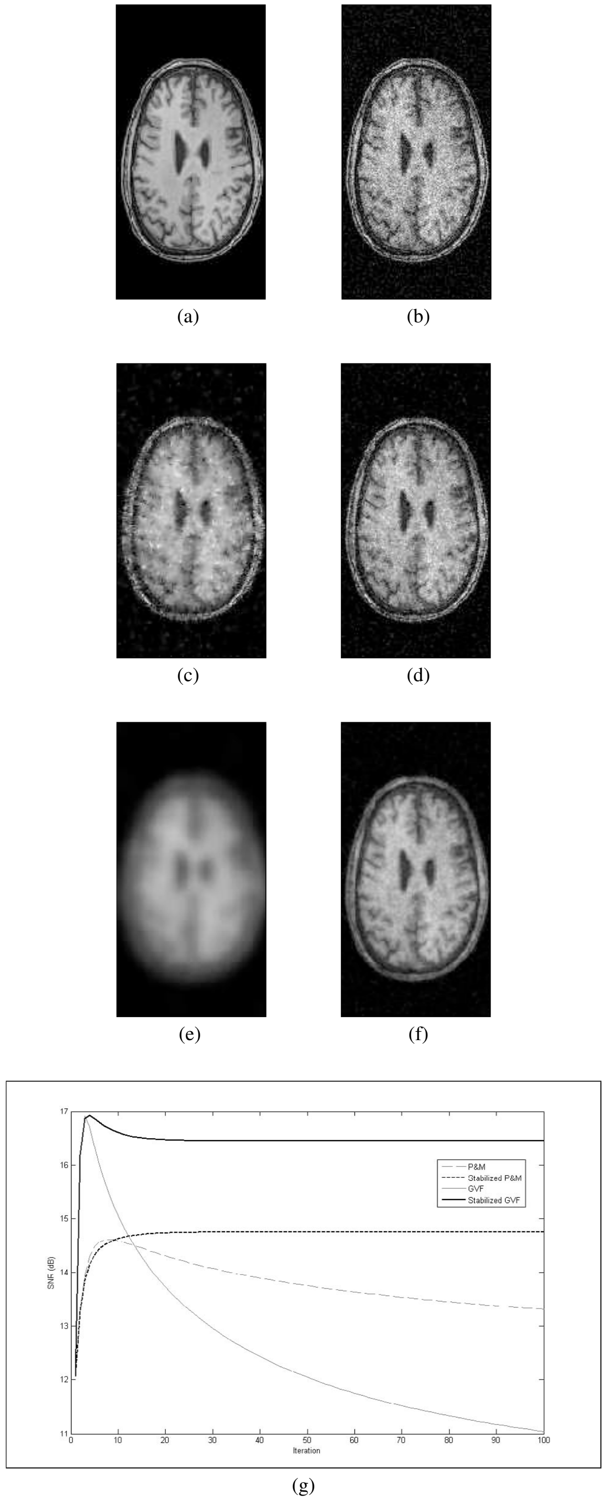

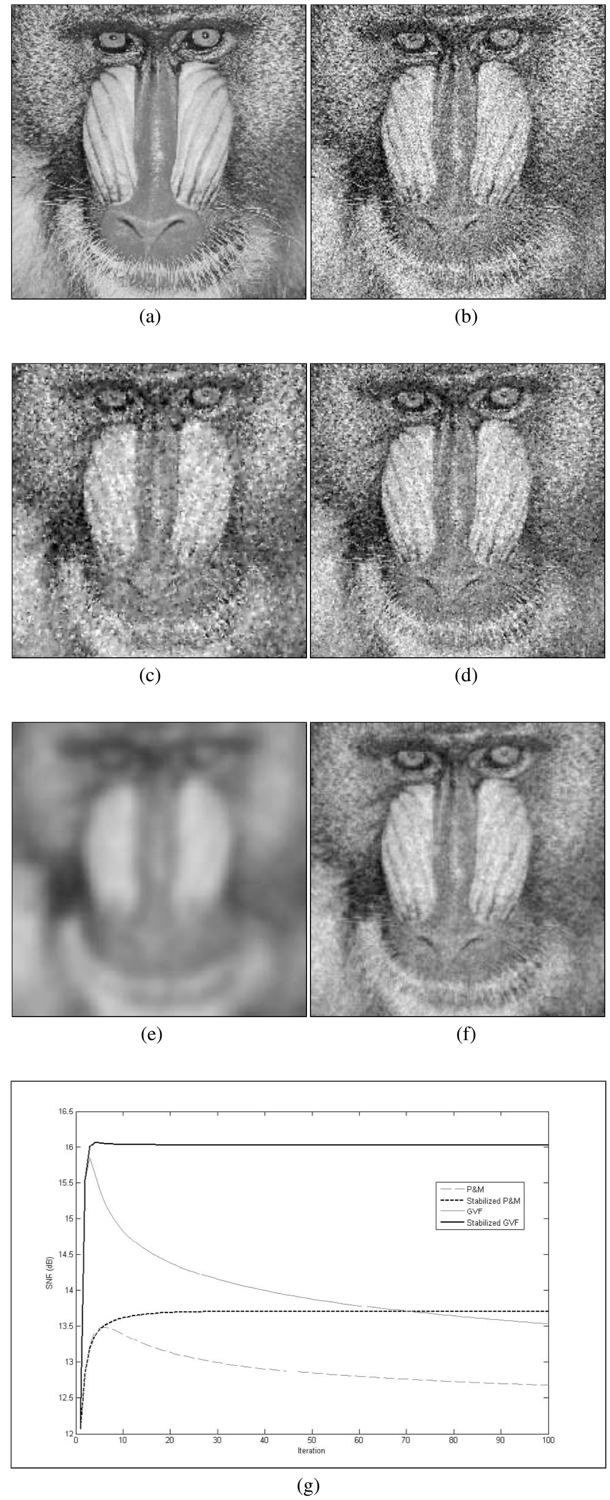

Fig. 1. Clinical MR image. (a) Original image. (b) Noisy image. (c) Result of the P&M algorithm. (d) Result of the stabilized P&M algorithm. (e) Result of the GVF algorithm. (f) Result of the stabilized P&M algorithm. (g) The SNR_t of different algorithms along the iterative smoothing process.

Fig. 2. Benchmark "Mandrill" image. (a) Original image. (b) Noisy image. (c) Result of the P&M algorithm. (d) Result of the stabilized P&M algorithm. (e) Result of the GVF algorithm. (f) Result of the stabilized P&M algorithm. (g) The SNR_t of different algorithms along the iterative smoothing process.