OVERSAMPLED INVERSE COMPLEX LAPPED TRANSFORM OPTIMIZATION

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ABSTRACT

When an oversampled FIR filter bank structure is used for signal analysis, a main problem is to guarantee its invertibility and to be able to determine an inverse synthesis filter bank. As the analysis scheme corresponds to a redundant decomposition, there is no unique inverse filter bank and some of the solutions can lead to artifacts in textured image filtering applications. In this paper, the flexibility in the choice of the inverse filter bank is exploited to find the best-localized impulse responses. The design is performed by solving a constrained optimization problem which is reformulated in a smaller dimensional space. Application to seismic data clearly shows the improvements brought by the optimization process.

***Index Terms*—** FIR digital filters, Transforms, Redundancy, Optimization methods, Seismic signal processing,

1. INTRODUCTION

Concepts of sparsity and redundancy have emerged as fundamental notions in the signal processing community. They are grounded on a redundant dictionary (instead of a basis) which is generally able to approximate a class of signals by the sum of a "small" number of atoms. One interesting subset of these overcomplete linear transforms consists of oversampled multirate filter banks (FBs). The latter possess advantages over classical critically sampled FBs. The first one is their improved robustness to noise and quantization. The second advantage lies in their more flexible design: on the analysis side, perfect reconstruction (PR) properties are less stringent in the oversampled case [1]; on the other side, synthesis filters, when they exist, are not unique in general. However, the appropriate design of inverse FIR filters remains complex. Recently, closed-form optimal expressions were obtained with 50% oversampled window DFT [2]. Filter banks appear under different names in image processing. Their early use was often limited to block transforms, such as the Hadamard or the Discrete Cosine Transform (DCT), popularized by the JPEG compression format. Its blocking or checkerboard artifacts at low bit-rates result from a relatively independent processing of adjacent blocks. These annoying effects have promoted the advent of lapped transforms (LTs) [3] and wavelets, which generally overlap. While wavelets are still popular (replacing DCT in the JPEG 2000 standard), LTs regain favour in signal processing [4], with for instance aliasing reduction using complex transforms, or the recent announcement of the H. D. Photo format based on a biorthogonal LT. In [5], we have proposed a method to guarantee that a given FIR analysis FB can be associated with an FIR synthesis FB with PR property and we have devised an algorithm to compute such inverse filters. Its application to DFT based FBs was utilized for 3D seismic data directional filtering. While generally providing nice visual results, blocking artifacts, interfering with automated interpretation, could result from an inaccurate choice of filtering parameters. This work contributes to an optimized design of the inverse filters of an oversampled FB, based on their time/space-localization. The solutions obtained by this originally constrained minimization problem, re-formulated as an unconstrained one, drastically reduce the filtering sensitivity to shrinkage operations in the transform domain.

In Section 2, we first recall the polyphase representation of FBs and we express the calculation of an inverse synthesis FB through the solutions of an underdetermined linear system. In Section 3, we address the optimal design problem and reformulate it as an unconstrained minimization problem. In Section 4, the proposed optimization is applied to DFT FBs, and the practical sensitivity reduction of transformed domain processing is demonstrated for seismic data filtering.

2. NOTATIONS AND PROBLEM

2.1. Notations

We first recall polyphase representation notations. Figure 1 represents a 1D M-band filter bank structure. The signal $(x(n))_{n\in\mathbb{Z}}$ is decomposed using M filters with impulse responses: $(h_i)_{0\leq i<M}$, each one having finite length kN with $k \in \mathbb{N}^*$. A decimation by an integer N is then performed. From the LT viewpoint, we therefore have $k-1$ overlapping blocks of size N. The M outputs of the analysis FB are denoted by $(y_i(n))_{0\leq i<M}$. The overall redundancy of the transform is thus $M/N = k'$. We are interested in the oversampled

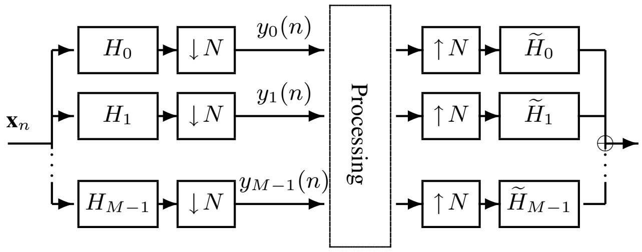

Fig. 1. Oversampled M-channel filter banks.

case, *i.e.* $k' > 1$.

With these notations, the outputs of the analysis FB are expressed, for all $i \in \{0, \dots, M-1\}$ and $n \in \mathbb{Z}$, as

$$y_i(n) = \sum_p h_i(p) x(Nn - p)$$
$$= \sum_\ell \sum_{j=0}^{N-1} h_i(N\ell + j) x(N(n-\ell) - j). \quad (1)$$

Let $\boldsymbol{H}(\ell) = (h_i(N\ell + j))_{0 \le i < M, 0 \le j < N}$, $\ell \in \{0, \cdots, k-1\}$ be the k matrices obtained from the impulse responses of the filters. We also define the polyphase vector signal from the input signal $x(n)$: $\forall n \in \mathbb{Z}$, $\mathbf{x}(n) = (x(Nn - j))_{0 \le j < N}$. A more concise form for Eq. (1) is:

$$\mathbf{y}(n) = (y_0(n), \dots, y_{M-1}(n))^\top$$
$$= \sum_\ell \boldsymbol{H}(\ell)\mathbf{x}(n-\ell) = (\boldsymbol{H} * \mathbf{x})(n),$$

or, equivalently, $\mathbf{y}[z] = \boldsymbol{H}[z]\mathbf{x}[z]$, where $\boldsymbol{H}[z] = \sum_{\ell=0}^{k-1} \boldsymbol{H}(\ell) z^{-\ell}$ is the $M \times N$ polyphase transfer matrix of the analysis filter bank and $\mathbf{x}[z]$ and $\mathbf{y}[z]$ are the z-transforms of $(\mathbf{x}(n))_{n \in \mathbb{Z}}$ and $(\mathbf{y}(n))_{n \in \mathbb{Z}}$, respectively. Similarly, we define the polyphase transfer matrix of the synthesis filter bank: $\widetilde{\boldsymbol{H}}[z] = \sum_\ell \widetilde{\boldsymbol{H}}(\ell) z^{-\ell}$ which is such that

$$\widetilde{\mathbf{x}}[z] = \widetilde{\boldsymbol{H}}[z]\mathbf{y}[z].$$

The polyphase vector of $(\widetilde{\mathbf{x}}(n))_{n \in \mathbb{Z}}$ is defined similarly as $(\mathbf{x}(n))_{n \in \mathbb{Z}}$ and

$$\widetilde{\boldsymbol{H}}(\ell) = \left(\widetilde{h}_j(N\ell - i)\right)_{0 \le i < N, 0 \le j < M}, \quad \ell \in \mathbb{Z}.$$

These expressions hold for any oversampled FB.

2.2. Problem statement

The goal is to achieve PR. In other words, we search a matrix $\widetilde{\boldsymbol{H}}[z]$ in $\mathbb{C}[z, z^{-1}]^{N \times M}$ such that $\widetilde{\boldsymbol{H}}[z]\boldsymbol{H}[z] = \boldsymbol{I}_N$. In our previous work [5], we have proposed a method to check whether a given FIR analysis FB can be inverted by an FIR synthesis FB: if $\boldsymbol{H}[z]$ is proven to be left invertible then there exists an integer p such that the polyphase transfer function of the synthesis FB reads: $\widetilde{\boldsymbol{H}}[z] = \sum_{\ell=1-p}^{0} \widetilde{\boldsymbol{H}}(\ell) z^{-\ell}$.

We obviously have

$$\widetilde{\boldsymbol{H}}[z]\boldsymbol{H}[z] = \sum_{\ell=1-p}^{0} \widetilde{\boldsymbol{H}}(\ell) z^{-\ell} \sum_{\ell=0}^{k-1} \boldsymbol{H}(\ell) z^{-\ell} = \sum_{\ell=1-p}^{k-1} \boldsymbol{U}(\ell) z^{-\ell},$$

where

$$\boldsymbol{U}(\ell) = \sum_{s=1+\max(\ell-k,-p)}^{\min(0,\ell)} \widetilde{\boldsymbol{H}}(s)\boldsymbol{H}(\ell - s).$$

The PR property is then equivalent to $\boldsymbol{U}(\ell) = \delta_\ell \boldsymbol{I}_N$, which leads to the following linear equation:

$$\mathcal{H}\widetilde{\mathcal{H}} = \mathcal{U} \quad (2)$$

where

$$\widetilde{\mathcal{H}}^\top = \left[\widetilde{\boldsymbol{H}}(1-p), \cdots, \widetilde{\boldsymbol{H}}(0)\right],$$
$$\mathcal{U}^\top = \left[\boldsymbol{0}_{N,(p-1)N}\ \boldsymbol{I}_N\ \boldsymbol{0}_{N,(k-1)N}\right],$$

and

$$\mathcal{H}^\top = \begin{pmatrix} \boldsymbol{H}(0) & \cdots & \boldsymbol{H}(k-1) & & 0 \\ & \ddots & & \ddots & \\ 0 & & \boldsymbol{H}(0) & \cdots & \boldsymbol{H}(k-1) \end{pmatrix}.$$

We have to solve the above system for increasing values of p in order to find the minimum order for an inverse polyphase transfer matrix.

3. OPTIMIZATION OF THE SYNTHESIS FB

We have provided conditions for a polyphase matrix to be associated to a PR synthesis FB. Since this system is underdetermined, we can exploit the remaining degrees of freedom to optimize the characteristics of the synthesis FB, namely their time/space localization.

3.1. Dimension reduction

Let r denote the rank of $\mathcal{H}$ and assume that $r < Mp$. We perform a Singular Value Decomposition (SVD) on this matrix:

$$\mathcal{H} = \mathcal{U}_0 \Sigma_0 \mathcal{V}_0^*,$$

where $\Sigma_0 \in \mathbb{C}^{r \times r}$ is an invertible diagonal matrix, $\mathcal{U}_0 \in \mathbb{C}^{N(k+p-1) \times r}$ and $\mathcal{V}_0 \in \mathbb{C}^{Mp \times r}$ are semi-unitary matrices. There exists therefore $\mathcal{U}_1 \in \mathbb{C}^{N(k+p-1) \times (N(k+p-1)-r)}$ and $\mathcal{V}_1 \in \mathbb{C}^{Mp \times (Mp-r)}$ such that $[\mathcal{U}_0\ \mathcal{U}_1]$ and $[\mathcal{V}_0\ \mathcal{V}_1]$ are unitary matrices. When an inverse polyphase transfer matrix exists, a particular solution to (2) is:

$$\widetilde{\mathcal{H}}^0 = \mathcal{H}^\sharp \mathcal{U},$$

where $\mathcal{H}^\sharp = \mathcal{V}_0\Sigma_0^{-1}\mathcal{U}_0^*$ is the pseudo-inverse of $\mathcal{H}$. Eq. (2) is then equivalent to:

$$\mathcal{U}_0\Sigma_0\mathcal{V}_0^*(\widetilde{\mathcal{H}} - \widetilde{\mathcal{H}}^0) = \boldsymbol{0}_{(N+k-1)\times N}.$$

Since $\mathcal{U}_0^*\mathcal{U}_0 = \boldsymbol{I}_r$ and Σ_0 is invertible, we get

$$\mathcal{V}_0^*(\widetilde{\mathcal{H}} - \widetilde{\mathcal{H}}^0) = \boldsymbol{0}_{r\times N},$$

which is equivalent to say that the columns of $\widetilde{\mathcal{H}} - \widetilde{\mathcal{H}}^0$ belong to the nullspace of $\mathcal{V}_0^*$, $\mathrm{Ker}(\mathcal{V}_0^*)$. Since $\mathrm{Ker}(\mathcal{V}_0^*)$ is equal to $\mathrm{Span}(\mathcal{V}_1)$, one can write:

$$\widetilde{\mathcal{H}} = \mathcal{V}_1\mathcal{C} + \widetilde{\mathcal{H}}^0 \tag{3}$$

where $\mathcal{C} \in \mathbb{C}^{(Mp-r)\times N}$. The design of the synthesis filter bank therefore reduces to the choice of $\mathcal{C}$. In our previous work, we took $\mathcal{C} = \boldsymbol{0}_{(Mp-r)\times N}$. However, as illustrated by Fig. 2-(a), this choice may result in synthesis filters with poor time-localization properties.

3.2. Optimal solution

To obtain impulse responses $(\widetilde{h}_j)_{0\le j<M}$ (for the synthesis FB) well-localized around some time-indices $(\overline{m}_j)_{0\le j<M}$, we propose to minimize the following cost function:

$$J(\widetilde{h}) = \sum_{j=0}^{M-1} \frac{\sum_m (m-\overline{m}_j)^2 \left|\widetilde{h}_j(m)\right|^2}{\sum_m \left|\widetilde{h}_j(m)\right|^2}$$

$$= \sum_{j=0}^{M-1} \frac{\sum_{\ell=1-p}^{0}\sum_{i=0}^{N-1}(\ell N - i - \overline{m}_j)^2 \left|\widetilde{H}_{i,j}(\ell)\right|^2}{\sum_{\ell=1-p}^{0}\sum_{i=0}^{N-1}\left|\widetilde{H}_{i,j}(\ell)\right|^2}.$$

Let us define, for all $\ell \in \{1-p,\dots,0\}$, $j \in \{0,\dots,M-1\}$ and $n \in \{0,\dots,Mp-r-1\}$

$$\boldsymbol{V}_j(\ell,n) = \mathcal{V}_1\big((\ell+p-1)M + j, n\big)$$

where $\mathcal{V}_1 = [\mathcal{V}_1(s,n)]_{0\le s<Mp, 0\le n<Mp-r}$. According to (3), we have:

$$\widetilde{H}_{i,j}(\ell) = \sum_{n=0}^{Mp-r-1} \boldsymbol{V}_j(\ell,n)\mathcal{C}(n,i) + \widetilde{H}^0_{i,j}(\ell)$$

where $\mathcal{C} = [\mathcal{C}(n,i)]_{0\le n<Mp-r, i\le r0<N}$. Let us introduce the matrices: $\boldsymbol{H}_j^0 = [\widetilde{H}^0_{i,j}(\ell)]_{1-p\le\ell\le 0, 0\le i\le N-1}$ and Λ_j defined by: $\Lambda_j = [(\ell N - i - \overline{m}_j)^2]_{1-p\le\ell\le 0, 0\le i\le N-1}$. Then we can write:

$$\widetilde{H}_{i,j}(\ell) = (\boldsymbol{V}_j\mathcal{C} + \boldsymbol{H}_j^0)_{\ell,i}.$$

Using the Frobenius norm:

$$\sum_{\ell=1-p}^{0}\sum_{i=0}^{N-1}\left|\widetilde{H}_{i,j}(\ell)\right|^2 = \left\|\boldsymbol{V}_j\mathcal{C} + \boldsymbol{H}_j^0\right\|^2$$

and the weighted Frobenius norm:

$$\sum_{\ell=1-p}^{0}\sum_{i=0}^{N-1}(\ell N - i - \overline{m}_j)^2\left|\widetilde{H}_{i,j}(\ell)\right|^2 = \left\|\boldsymbol{V}_j\mathcal{C} + \boldsymbol{H}_j^0\right\|^2_{\Lambda_j},$$

we deduce that:

$$J(\widetilde{h}) = \widetilde{J}(\mathcal{C}) = \sum_{j=0}^{M-1}\frac{\left\|\boldsymbol{V}_j\mathcal{C} + \boldsymbol{H}_j^0\right\|^2_{\Lambda_j}}{\left\|\boldsymbol{V}_j\mathcal{C} + \boldsymbol{H}_j^0\right\|^2}.$$

The constrained minimization problem is now re-expressed as the unconstrained minimization of $\widetilde{J}$.

4. APPLICATION AND RESULTS

4.1. Seismic data filtering

In [6] we showed complex-valued modulated filters were well-suited to texture-like seismic data. Indeed, these data present highly anisotropic features that are well captured by complex-valued transforms, and their oscillatory behaviour, due to the layered underground structure, is well described by an harmonic transform. The analysis filters we used are derived from [7] and expressed as: $h_i(n) = \boldsymbol{E}(i,n)h_a(n)$, where

$$\boldsymbol{E}(i,n) = \frac{1}{\sqrt{k'N}} e^{-\imath(i-\frac{k'N}{2}+\frac{1}{2})(n-\frac{kN}{2}+\frac{1}{2})\frac{2\pi}{k'N}},$$

and $(h_a(n))_{1\le n\le kN}$ is a non vanishing analysis window such as:

$$h_a(n) = \sin\Big(\frac{n\pi}{kN+1}\Big).$$

Seismic data are typically two or three dimensional. By applying the above monodimensional transform separably in all directions we define a multi-dimensional transform.

Unwanted directional structures, due to pre-processing or physical perturbations during data acquisition for instance, may corrupt seismic data and hinder subsequent automated interpretation. In a nutshell, to perform a directional filtering retaining features of interest, we first compute the locally dominant direction, then filter out coefficients not corresponding to orientations close to the dominant one, and finally use thresholding to remove small coefficients likely to represent noise. The appropriate choice of the threshold may constitute an issue to the filtering robustness.

4.2. Optimal Synthesis Filter Bank

In Section 3 we have reduced the dimension of the problem from MpN to $(Mp-r)N$. In practice, the rank r is large; the dimension reduction is therefore significant to the computational burden of the synthesis. As an illustration we have tested both the constrained and the unconstrained optimizations, with $N = 8$, $k = 3$ and $k' = 7/4$, on a 3GHz Pentium 4 processor. The computation of the optimal inverse took 247 seconds for the constrained problem, compared to 5 seconds with unconstrained optimization, including both the SVD and the minimization; the re-formulation of the problem thus leads to a significant computational gain.

Figure 2 represents the magnitude of the impulse responses for 4 of the $M = k'N = 14$ synthesis filters in the pseudo inverse and the optimal case. Coefficients in the first case are quite scattered with large border coefficients (potentially causing blocking artifacts), while in the second case the coefficient distribution seems more window shaped, better clustered around the center of the impulse responses.

4.3. Results on real data

For this application, the parameters are set $N = 16$, $k' = 7/4$ and $k = 3$. Using the method described in Subsection 2.2, it is found that $p = 3$. We have chosen processing parameters (the tolerance on the retained directions and the threshold) to target a situation in which the reconstruction leads to poor visual results. Figure 3 presents real seismic data which are filtered and then reconstructed using both the pseudo-inverse FB and the optimal one. On Fig. 3-(c), a grid pattern appears in the pseudo-inverse case, due to high value of the threshold and the poor impulse responses of the inverse filters. The second filtered image (d) and the difference image (b) show clearly that this pattern, without any geological meaning, is strongly attenuated with the optimal FB, leading to better visual results.

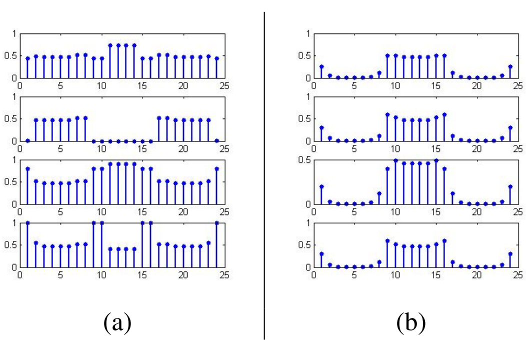

Fig. 2. Examples of the magnitude of the impulse responses in (a) the pseudo-inverse case and (b) after optimization.

5. CONCLUSIONS

By taking advantage of the degrees of freedom offered by oversampled FBs, we have proposed a method to design optimally spatially-localized synthesis FBs. Our study allows us to reduce the design problem to an unconstrained optimization which can be solved quite efficiently by numerical methods. The resulting optimized FBs have been shown to be appropriate for seismic data filtering. In our future work, we plan to consider more general forms of cost functions.

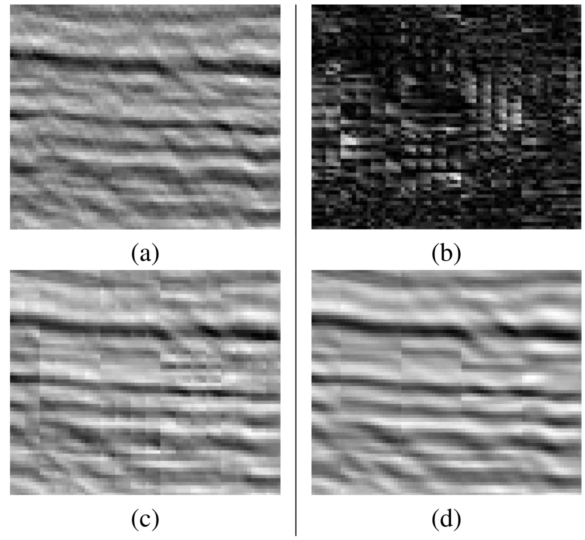

Fig. 3. (a) Seismic image, (b) Difference between the two filtered results, (c) Filtered with pseudo inverse FB (d) Filtered using optimal FB.

6. REFERENCES

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