PERFORMANCE ANALYSIS OF THE FXLMS-BASED NARROWBAND ACTIVE NOISE CONTROL SYSTEM WITH ONLINE SECONDARY PATH MODELING

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ABSTRACT

Rotating machines such as diesel engines, cutting machines, fans, etc. generate sinusoidal noise signals that may be successfully reduced by narrowband active noise control (ANC) systems. In this paper, performance analysis of such a typical filtered-X LMS (FXLMS) based narrowband ANC system equipped with an online secondary path modeling subsystem is conducted in detail. First, difference equations governing the dynamics of the FXLMS for secondary source synthesis and the LMS for secondary path estimation are derived in terms of convergence of both mean and mean square. Steady-state expressions for mean square error (MSE) as well as the remaining noise power are then developed in closed forms. Extensive simulations are performed to demonstrate the validity of the analytical results.

***Index Terms*—** Narrowband active noise control, Online secondary path modeling, Performance analysis, Convergence, Steady-state mean square error (MSE)

1. INTRODUCTION

Rotating machines such as diesel engines, cutting machines, fans, etc. generate noise signals that can usually be modeled as sinusoidal signals in additive noise. Suppressing or reducing the noise signals, especially their lower frequency portion, is very important in various engineering and environmental systems. Narrowband active noise control (ANC) systems are designated to remove or mitigate these annoying noise sources [1]-[7].

A large number of ANC systems have been proposed in the literature, and some of them have been implemented in real-life applications [3]. Usually, the finite-impulse-response (FIR) filters adapted by a filtered-X least mean square (FXLMS) algorithm or its variants are applied [3]. Other techniques using recursive least squares (RLS) and Kalman filtering based algorithms have also been developed for many ANC systems [6, 3], which generally provide better noise reduction performance at the expense of more computational cost. In most ANC systems, the secondary path is assumed to be known *a prior*, which is estimated in some way in advance [3]. Recently, much attention has been paid to the online secondary path modeling techniques which allow the secondary path to be estimated in an online fashion such that the secondary path is identified and utilized while the system is in operation [3].

The conventional narrowband ANC systems are effective in suppressing sinusoidal noise sources in many real-life applications [3]. Fig.1 shows such a conventional ANC system [3, 4] with online secondary modeling using auxiliary noise. Statistical properties of this ANC system with known secondary path has been analyzed in some detail [8]. But the same ANC system equipped with secondary path modeling has not been investigated. Two adaptive subsystems in Fig.1, one synthesizing the secondary source and the other estimating the secondary path, are cascaded in a way that makes it difficult to assess the performance of the entire system.

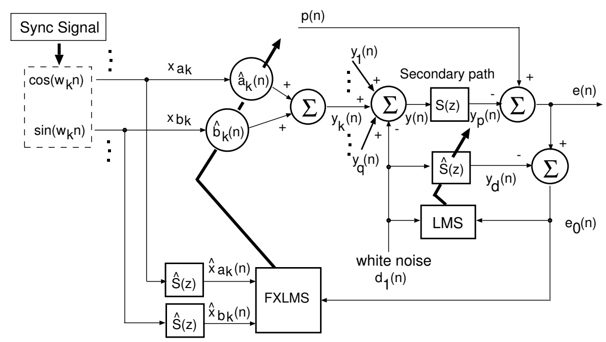

Fig. 1 Block diagram of a conventional narrowband ANC system with secondary path modeling (the i-th channel).

In this paper, performance analysis of the FXLMS-based ANC system with online secondary path modeling is performed in detail. Difference equations governing the dynamics of the system are developed in terms of estimation errors between the discrete Fourier coefficients (DFCs) estimates of the secondary source and their optimal values which assure perfect cancellation for all the primary sinusoids being targeted, and between the secondary path estimate and its actual value. The steady-state estimation mean squared errors (MSEs) as well as the remaining noise power are also de-

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rived in closed forms. Stability of the system is also discussed where possible. Extensive simulations are performed to illustrate the validity of the analytical results. The formulation of the problem is now provided below.

The primary noise signal in Fig. 1 is given by

$$p(n) = \sum_{i=1}^{q} \{a_i \cos(\omega_i n) + b_i \sin(\omega_i n)\} + v_p(n) \quad (1)$$

where q is the number of frequency components of the sinusoidal signal, ω_i is the frequency of the i-th component, $v_p(n)$ is a zero-mean additive white Gaussian noise with variance σ_p^2. The signal frequencies may be identified in a regression fashion based on a synchronization (sync) signal derived from a non-acoustical sensor like a tachometer.

The secondary source generated by the synthesis subsystem is expressed by

$$y_o(n) = \sum_{i=1}^{q} y_i(n) = \sum_{i=1}^{q} \left\{ \hat{a}_i(n) x_{a_i}(n) + \hat{b}_i(n) x_{b_i}(n) \right\} \quad (2)$$

$$x_{a_i}(n) = \cos(\omega_i n), \;\; x_{b_i}(n) = \sin(\omega_i n) \quad (3)$$

The FXLMS algorithm for DFC estimates is given by

$$\hat{a}_i(n+1) = \hat{a}_i(n) + \mu_i e(n) \hat{x}_{a_i}(n) \quad (4)$$

$$\hat{b}_i(n+1) = \hat{b}_i(n) + \mu_i e(n) \hat{x}_{b_i}(n) \quad (5)$$

where

$$e(n) = p(n) - S(z) y(n), \;\; y(n) = \sum_{k=1}^{q} y_k(n) - d_1(n) \quad (6)$$

$$\hat{x}_{a_i}(n) = \hat{S}(z,n) x_{a_i}(n) = \hat{\alpha}_i(n) x_{a_i}(n) + \hat{\beta}_i(n) x_{b_i}(n) \quad (7)$$

$$\hat{x}_{b_i}(n) = \hat{S}(z,n) x_{b_i}(n) = -\hat{\beta}_i(n) x_{a_i}(n) + \hat{\alpha}_i(n) x_{b_i}(n) \quad (8)$$

$$S(z) = \sum_{j=0}^{M-1} s_j z^{-j}, \;\; \hat{S}(z,n) = \sum_{j=0}^{\hat{M}-1} \hat{s}_j(n) z^{-j} \quad (9)$$

$$\hat{\alpha}_i(n) = \sum_{j=0}^{\hat{M}-1} \hat{s}_j(n) \cos(j\omega_i), \hat{\beta}_i(n) = \sum_{j=0}^{\hat{M}-1} \hat{s}_j(n) \sin(j\omega_i) \quad (10)$$

and μ_i is a step size parameter for the i-th channel. $S(z)$ is the true secondary path, $\hat{S}(z,n)$ is an estimate of $S(z)$ at time instant n, which is obtained online by the LMS-based secondary path estimation subsystem. The parameters M and $\hat{M}$ are the system orders of the true and estimated secondary paths, respectively, and $d_1(n)$ is an auxiliary white noise with zero mean and variance $\sigma_{d_1}^2$. The LMS algorithm for the secondary path estimation is given by

$$\hat{s}_m(n+1) = \hat{s}_m(n) + \mu_{s,m} e_0(n) d_1(n-m) \quad (11)$$

$$e_0(n) = e(n) - y_d(n), \;\; y_d(n) = \hat{S}(z,n) d_1(n) \quad (12)$$

where $\mu_{s,m}$ is a step size parameter for the m-th coefficient.

2. PERFORMANCE ANALYSIS

Using (2) and (7)-(10) in $e(n)$ yields

$$\begin{aligned} e(n) \approx \sum_{i=1}^{q} \Big\{ & [a_i - (\alpha_i \hat{a}_i(n) - \beta_i \hat{b}_i(n))] x_{a_i}(n) \\ & + [b_i - (\beta_i \hat{a}_i(n) + \alpha_i \hat{b}_i(n))] x_{b_i}(n) \Big\} \\ & + v_p(n) + \sum_{j=0}^{M-1} s_j d_1(n-j) \end{aligned} \quad (13)$$

where $\hat{a}_i(n-j) \approx \hat{a}_i(n)$, $\hat{b}_i(n-j) \approx \hat{b}_i(n)$ for $j = 1, 2, \cdots, M-1$ are used to facilitate and simplify the analysis that follows. Based on extensive numerical simulations it is revealed that the above does not affect the accuracy of analysis significantly even for relatively fast adaptation (see simulation results in Section 3). Obviously, from (13), the optimum DFCs that assure a perfect cancellation of all the sinusoids satisfy

$$\begin{bmatrix} a_{i,opt} \\ b_{i,opt} \end{bmatrix} = \begin{bmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i \\ b_i \end{bmatrix} \quad (14)$$

$$\alpha_i = \sum_{j=0}^{M-1} s_j \cos(j\omega_i), \;\; \beta_i = \sum_{j=0}^{M-1} s_j \sin(j\omega_i) \quad (15)$$

Define the estimation errors of DFCs as

$$\varepsilon_{a_i}(n) = a_{i,opt} - \hat{a}_i(n), \;\; \varepsilon_{b_i}(n) = b_{i,opt} - \hat{b}_i(n) \quad (16)$$

Eventually, the error signal reduces to

$$\begin{aligned} e(n) \approx \sum_{i=1}^{q} \{ & [\alpha_i \varepsilon_{a_i}(n) - \beta_i \varepsilon_{b_i}(n)] x_{a_i}(n) \\ & + [\beta_i \varepsilon_{a_i}(n) + \alpha_i \varepsilon_{b_i}(n)] x_{b_i}(n) \} + v_p(n) + \sum_{j=0}^{M-1} s_j d_1(n-j) \end{aligned} \quad (17)$$

A. Convergence in the mean sense

Using the above error signal expression in the FXLMS recursions (4), (5) and the LMS update (11), and taking ensemble average ($E[\cdot]$) yields

$$\begin{aligned} E[\varepsilon_{a_k}(n+1)] & \\ = & \left\{ 1 - \frac{1}{2}\mu_k (\alpha_k E[\hat{\alpha}_k(n)] + \beta_k E[\hat{\beta}_k(n)]) \right\} E[\varepsilon_{a_k}(n)] \\ & - \frac{1}{2}\mu_k (\alpha_k E[\hat{\beta}_k(n)] - E[\hat{\alpha}_k(n)]\beta_k) E[\varepsilon_{b_k}(n)] \end{aligned} \quad (18)$$

$$\begin{aligned} E[\varepsilon_{b_k}(n+1)] & \\ = & \left\{ 1 - \frac{1}{2}\mu_k (\alpha_k E[\hat{\alpha}_k(n)] + \beta_k E[\hat{\beta}_k(n)]) \right\} E[\varepsilon_{b_k}(n)] \\ & - \frac{1}{2}\mu_k (-\alpha_k E[\hat{\beta}_k(n)] + E[\hat{\alpha}_k(n)]\beta_k) E[\varepsilon_{a_k}(n)] \end{aligned} \quad (19)$$

$$E[\varepsilon_{s,m}(n+1)] = (1 - \mu_{s,m}\sigma_{d_1}^2) E[\varepsilon_{s,m}(n)] \quad (20)$$

where

$$E[\hat{\alpha}_k(n)] = -\sum_{j=0}^{\hat{M}-1} E[\varepsilon_{s,j}(n)]\cos(j\omega_k) + \sum_{j=0}^{\hat{M}-1} s_j \cos(j\omega_k) \quad (21)$$

and $E[\hat{\beta}_k(n)]$ can be calculated similarly. The estimation error of the secondary path is defined by $E[\varepsilon_{s,m}(n)] = E[\hat{s}_m(n) - s_m]$. Clearly, in the mean sense convergence of the LMS is independent of the FXLMS and is guaranteed as long as $0 < \mu_{s,m} < 2/\sigma^2_{d_1}$. The cosine and sine DFC errors will converge independently if $\hat{S}(z,n)$ becomes very close to its true value. A stability bound for μ_k may be obtained by letting the absolute eigenvalues of (18) and (19) equal to unity, as

$$\mu_{k,bound}(n) = \quad (22)$$

$$\frac{4\left\{\alpha_k E[\hat{\alpha}_k(n)] + \beta_k E[\hat{\beta}_k(n)]\right\}}{\left\{\alpha_k E[\hat{\alpha}_k(n)] + \beta_k E[\hat{\beta}_k(n)]\right\}^2 + \left\{-\alpha_k E[\hat{\beta}_k(n)] + E[\hat{\alpha}_k(n)]\beta_k\right\}^2}$$

which is the same as the so-called $-90°$ condition [9]. In the above derivations, $x_{a_i}(n)$ and $x_{b_i}(n)$ are treated as pseudo-random noises [1, 7].

B. Convergence in the mean square sense

Putting (16) into (4) and squaring both sides, one gets

$$E[\varepsilon^2_{a_k}(n+1)] = E[\varepsilon^2_{a_k}(n)] \quad (23)$$
$$-2\mu_k \underbrace{E[\varepsilon_{a_k}(n)e(n)\hat{x}_{a_k}(n)]}_{I_k(n)} + \mu_k^2 \underbrace{E[e^2(n)\hat{x}^2_{a_k}(n)]}_{K_k(n)}$$

After some technical manipulations, one gets

$$I_k(n) = \gamma_{a_k}(n) \quad (24)$$
$$+\frac{1}{2}\{\alpha_k E[\hat{\alpha}_k(n)] + \beta_k E[\hat{\beta}_k(n)]\}E[\varepsilon^2_{a_k}(n)]$$
$$+\frac{1}{2}\{-\beta_k E[\hat{\alpha}_k(n)] + \alpha_k E[\hat{\beta}_k(n)]\}E[\varepsilon_{a_k}(n)]E[\varepsilon_{b_k}(n)]$$

where

$$\gamma_{a_k}(n) = E\left[\varepsilon_{a_k}(n)\sum_{j=0}^{M-1} s_j d_1(n-j)\hat{x}_{a_k}(n)\right]$$
$$= -\frac{1}{2}\mu_k \sum_{m=1}^{M-1} [\{E[\hat{\alpha}_k(n)]E[\hat{\alpha}_k(n-m)]\cos(m\omega_k)$$
$$-E[\hat{\alpha}_k(n)]E[\hat{\beta}_k(n-m)]\sin(m\omega_k)$$
$$+E[\hat{\beta}_k(n)]E[\hat{\alpha}_k(n-m)]\sin(m\omega_k)$$
$$+E[\hat{\beta}_k(n)]E[\hat{\beta}_k(n-m)]\cos(m\omega_k)\}$$
$$\times \sum_{j_1=0}^{M-1}\sum_{j_2=0}^{M-1} s_{j_1}s_{j_2}\sigma^2_{d_1}\delta(m+j_1-j_2)] \quad (25)$$

where $\delta(\cdot)$ is a Dirac delta function. It has been found that the evaluation of (25) affects the analysis accuracy considerably. This is due to the fact that the correlation between $\varepsilon_{a_k}(n)$ and $\sum_{j=0}^{M-1} s_j d_1(n-j)\hat{x}_{a_k}(n)$, seemingly negligible, is surprisingly significant such that neglecting this results in large discrepancy between analysis and simulations.

Next, after some lengthy manipulations, one gets

$$K_k(n) = \sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma^2_{d_1} + \frac{1}{2}E[\hat{\alpha}^2_k(n)] + \frac{1}{2}E[\hat{\beta}^2_k(n)] \quad (26)$$
$$+\frac{3}{8}E[\hat{\alpha}^2_k(n)]E[(\alpha_k\varepsilon_{a_k}(n) - \beta_k\varepsilon_{b_k}(n))^2]$$
$$+\frac{1}{4}E[\hat{\alpha}^2_k(n)]\sum_{i=1,i\neq k}^{q} E[(\alpha_i\varepsilon_{a_i}(n) - \beta_i\varepsilon_{b_i}(n))^2]$$
$$\vdots \;\; (omitted\ due\ to\ space\ limitation)$$

where $E[\hat{\alpha}^2_k(n)], E[\hat{\beta}^2_k(n)], E[\hat{\alpha}_k(n)\hat{\beta}_k(n)]$ may be properly evaluated (details omitted). Following along the same lines, similar difference equations can be derived for $E[\varepsilon^2_{b_k}(n)]$. The difference equations for the secondary path estimation errors are derived as follows

$$E[\varepsilon^2_{s,m}(n+1)] = \{1 - 2\mu_{s,m}\sigma^2_{d_1} + 3\sigma^4_{d_1}\mu^2_{s,m}\}E[\varepsilon^2_{s,m}(n)]$$
$$+\mu^2_{s,m}\sigma^4_{d_1}\sum_{j=0,j\neq m}^{\hat{M}-1} E[\varepsilon^2_{s,j}(n)] + \mu^2_{s,m}\sigma^2_p\sigma^2_{d_1} \quad (27)$$
$$+\frac{1}{2}\mu^2_{s,m}\sum_{i=1}^{q}\{(\alpha_i^2 + \beta_i^2)(E[\varepsilon^2_{a_i}(n)] + E[\varepsilon^2_{b_i}(n)])\}\sigma^2_{d_1}$$

Now, the mean and the mean square senses convergences as well as the steady-state properties of the system may be evaluated by solving the difference equations (18)-(20), (23) and (27) simultaneously, even though they are highly nonlinear. Stability bounds tighter than (23) for the step size parameters can also be determined numerically by grid search. But closed-form expressions are infeasible due to nonlinearity.

C. Steady-state MSE expressions

When the system reaches its steady state, all the mean estimation errors will converge to zeros, as can be proved and deduced from (18)-(20). We have shown that $E[\varepsilon^2_{a_k}(n)]|_{n\to\infty}$ is identical to $E[\varepsilon^2_{b_k}(n)]|_{n\to\infty}$ (proof omitted). Now, let

$$E[\varepsilon^2_{a_k}(n)]|_{n\to\infty} = J_k(\infty), E[\varepsilon^2_{s,m}(n)]|_{n\to\infty} = J_{s,m}(\infty) \quad (28)$$

The steady-state MSEs and the remaining noise power are eventually derived as follows

$$J_k(\infty) = \mu_k\eta_k \quad (29)$$

$$J_{s,m}(\infty) = \frac{\mu_{s,m}(1 + \sigma^2_{d_1}\phi_s)(\sigma_p^2 + \xi)}{2 - \mu_{s,m}\sigma^2_{d_1}} \quad (30)$$

$$E[e^2(\infty)] = \xi + \sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma^2_{d_1} \quad (31)$$

where

$$\eta_k = R_k(\infty) + \frac{1}{2}(\sigma_p^2 + \sum_{j=0}^{M-1} s_j^2 \sigma_{d_1}^2)\left(1 + \frac{\phi_s \sigma_p^2}{\alpha_k^2 + \beta_k^2}\right) \quad (32)$$

$$+ \frac{\left\{\phi_s(\sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma_{d_1}^2) + (\alpha_k^2 + \beta_k^2 + \phi_s\sigma_p^2)\right\}\xi}{2(\alpha_k^2 + \beta_k^2)} + \frac{\phi_s \xi^2}{2(\alpha_k^2 + \beta_k^2)}$$

$$\xi = A/B \quad (33)$$

$$A = \sum_{k=1}^{q} \mu_k(\alpha_k^2 + \beta_k^2)R_k(\infty) + \frac{1}{2}(\sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma_{d_1}^2) \times \sum_{k=1}^{q}\mu_k(\alpha_k^2 + \beta_k^2) + \frac{1}{2}\phi_s\sigma_p^2(\sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma_{d_1}^2)\sum_{k=1}^{q}\mu_k$$

$$B = 1 - \frac{1}{2}\phi_s\sigma_p^2\sum_{k=1}^{q}\mu_k - \frac{1}{2}\sum_{k=1}^{q}\mu_k(\alpha_k^2 + \beta_k^2) - \frac{1}{2}\phi_s(\sigma_p^2 + \sum_{j=0}^{M-1} s_j^2\sigma_{d_1}^2)\sum_{k=1}^{q}\mu_k$$

$$\phi_s = \frac{\rho}{1-\rho},\ \rho = \sum_{m=0}^{\hat{M}-1}\frac{\mu_{s,m}}{2(1-\mu_{s,m}\sigma_{d_1}^2)} \quad (34)$$

It may be concluded that 1) the MSEs of DFCs and secondary path, (29) and (30), are both approximately proportional to their step size parameters if the adaptation is sufficiently slow, but these MSEs are related to each other in a very complicated way through variables ξ and ϕ_s which are determined by the step sizes, the signal parameters and the secondary path, and 2) the remaining noise power consists of three parts; remaining frequency components due to the misadjustment of the two subsystems, the variance of additive noise $v_p(n)$, and the power contributed by the auxiliary noise $d_1(n)$.

3. SIMULATIONS

Extensive simulations are performed to demonstrate and show the validity of the derived difference equations and steady-state MSE expressions. Very good agreements are observed between analytical results and simulations. A representative comparison between theory and simulation is given in Fig.2.

4. CONCLUSIONS

In this paper, the conventional FXLMS-based narrowband ANC system with LMS online secondary path modeling has been investigated in detail. Simulations are conducted to show the validity of the analytical findings that significantly enhance our understanding of the system's behavior.

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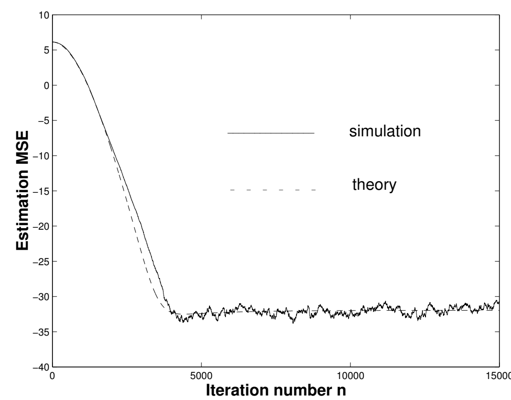
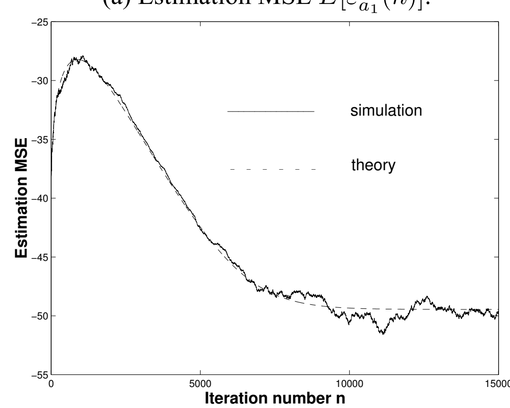


(a) Estimation MSE $E[\varepsilon_{a_1}^2(n)]$.



(b) Estimation MSE $E[\varepsilon_{s_1}^2(n)]$.

Fig. 2 Comparisons between theory and simulations (signal frequency: $0.10\pi,\ 0.20\pi,\ 0.30\pi$; $\mathbf{a} = [2.0\ 1.0\ 0.5]^T$, $\mathbf{b} = [-1.0\ -0.5\ 0.1]^T$; uniform step sizes: $\mu_i = 0.005$, $\mu_{s,m} = 0.002$; $\sigma_p = 0.10$, $\sigma_{d_1} = 0.50$; $M = \hat{M} = 11$, 100 runs).