A PHYSICAL MODEL FOR PLATE REVERBERATION

Stefan Bilbao

Sonic Arts Research Centre
Queen's University Belfast

Kevin Arcas, Antoine Chaigne

Unité de Mécanique - ENSTA
Palaiseau, France

ABSTRACT

In this article, a digital plate reverberation algorithm is presented, based on a direct numerical simulation of the equations of motion of a thin linear plate of Kirchhoff type. While such an algorithm will be more expensive, computationally, than digital filter-based algorithms, the resulting algorithm allows far more flexible control on the part of the user, in that the defining parameters have physical significance (i.e., they are related directly to material and geometry of the plate itself). A partial differential equation model is presented, followed by a discussion of a finite difference scheme, which is then specialized to the case of plate reverberation; numerical simulation results are presented.

1. INTRODUCTION

Physical modeling techniques have, in the past twenty years, been increasingly employed for sound synthesis; the benefits of such an approach are manifold: in addition to producing extremely high-quality audio output, often rendering some of the more subtle timbral nuances of acoustic instruments, they allow a very flexible means of control, in that the parameters which define such algorithms have physical significance. There are many such techniques, all of which take their final form as digital algorithms: digital waveguides [9] , modal synthesis [1] and direct numerical simulation techniques such as finite difference schemes [4] are the most well known. It is also possible, however, to create physical models not just of musical instruments, but also mechanically-based audio effect units. A prime example of such an effect is the well-known plate reverberation unit, which had its heyday in the 1960s, and has since become a classic effect. In this paper, the vibrations of the plate resonator are numerically simulated using a finite difference scheme. As mentioned above, this allows for very flexible control and tuning of the resonator, including variation in not only the plate geometry and thickness, but also its material and damping rates. In addition, it becomes possible to explore possibilities which are not easily realized in a physical set-up, including multiple moving input points and pickups.

In Section 2, a simple model of plate vibration is presented, followed by a finite difference scheme in Section 3 and a discussion of various aspects, including boundary termination, and input/output issues. In Section 4, a numerical example of plate reverberation is presented.

2. A PLATE MODEL

In its simplest form, a plate reverberation unit may be modelled as a thin flat rectangular plate undergoing small-amplitude vibration; such a plate is well-described by the following partial differential equation (PDE) [2] :

$$\frac{\partial^2 w}{\partial t^2} = -\kappa^2\nabla^4 w + c^2\nabla^2 w - 2\sigma\frac{\partial w}{\partial t} + b\frac{\partial}{\partial t}\nabla^2 w + f(x_{in}, y_{in}, t), \quad (1)$$

which is the classical Kirchhoff model [6, 11], accompanied by several additional terms, the significance of which are described below. It can also be thought of as a generalization to two dimensions of the so-called "stiff string" model, commonly encountered in physical modelling for string instruments. Equation (1) is defined for $t \geq 0$, over the region $x \in [0, L_x]$, $y \in [0, L_y]$. ∇^2 is the Laplacian, and ∇^4 the biharmonic operator. $w(x, y, t)$ represents the transverse plate deflection, and the stiffness parameter κ^2 is defined by

$$\kappa^2 = \frac{Eh^2}{12\rho(1-\nu^2)},$$

where E, h, ρ and ν are Young's modulus, plate thickness, density and Poisson's ratio, respectively for the plate. The term involving the parameter c, though it does not often appear in plate models, represents the contribution due to tensioning in the plate reverberation unit.

Damping in metallic plates mainly comes from thermoelastic and radiation losses [5]. Modelling such mechanisms is algebraicaly complex and requires parameter validation through experiment. We have chosen to use, for simplicity, a model analogous to that typically used for string damping. The dispersion relation of the plate model (2) can be formulated as a relation between the real wavenumber γ and the complex frequency $\omega = \Omega + j\alpha$, where α is commonly called the loss factor.

$$\omega^2 = \kappa^2\gamma^4 + c^2\gamma^2 + 2\sigma j\omega + bj\omega\gamma^2. \qquad (2)$$

At high frequencies the tension term can be neglected, and the loss factor, which determines the exponential time decay of the vibrations, is given by

$$\alpha(\omega) \approx \sigma + \frac{b}{2\kappa}\omega. \qquad (3)$$

The term of coefficient 2σ in (2) controls the overall decay rate of plate displacement and the term with coefficient b allows for increasing rates of losses at high frequencies.

The term $f(x_{in}, y_{in}, t)$ is of particular importance in the context of plate reverberation in that it represents the source or input waveform to be processed. Using a source of the form $\delta(t)\delta(x - x_{in})\delta(y - y_{in})$ gives the impulse response of the reverberation unit, for a source at a given position.

The second order time-dependent PDE (1) requires the specification of two conditions at any boundary; the only two conditions we will examine in this short paper are the so-called free conditions, given by

$$\frac{\partial^2 w}{\partial x_n^2} + \nu\frac{\partial^2 w}{\partial x_s^2} = \frac{\partial^3 w}{\partial x_n^3} + (2-\nu)\frac{\partial^3 w}{\partial x_s^2 \partial x_n} = 0, \qquad (4)$$

where $\frac{\partial}{\partial x_n}$ and $\frac{\partial}{\partial x_s}$ represent partial derivatives in a direction normal to and tangential to a given boundary, respectively.

3. A FINITE DIFFERENCE SCHEME

An explicit finite difference scheme for system (1) may be arrived at in a straightforward way through difference approximations to the various differential operators, and by replacing the dependent variable $w(x, y, t)$ by a grid function $w_{i,j}^n$, representing an approximation to w at coordinates $x = i\Delta$, $y = j\Delta$ and $t = nT$, for integer i, j and n. T is the time step and Δ is the spacing between adjacent grid points. Due to stability considerations, these quantities may not be chosen independently; see Section 3.1 for further comments.

A suitable difference scheme has been derived and presented, in the context of sound synthesis, in [2], and may be simply written as a recursion in the grid function $w_{i,j}^n$, where an interior point in the domain is updated according to

$$\begin{aligned} w_{i,j}^{n+1} &= \eta \sum_{|k|+|l|\leq 2} \beta_{|k|,|l|} w_{i+k,j+l}^n \qquad (5) \\ &+ \eta \sum_{|k|+|l|\leq 1} \gamma_{|k|,|l|} w_{i+k,j+l}^{n-1} + T^2 \hat{f}_{i,j}^n, \end{aligned}$$

where $\hat{f}_{i,j}^n$ is an approximation to the dry source signal $f(i\Delta, j\Delta, nT)$ and

$$\begin{aligned} \beta_{0,0} &= 2 - 20\mu^2 - 4(\lambda^2 + \xi), \\ \beta_{1,0} &= \beta_{0,1} = 8\mu^2 + \lambda^2 + \xi, \\ \beta_{1,1} &= -2\mu^2, \\ \beta_{2,0} &= \beta_{0,2} = -\mu^2, \\ \gamma_{0,0} &= -1 + 4\xi + \sigma T, \\ \gamma_{1,0} &= \gamma_{0,1} = -\xi, \end{aligned}$$

and where we have also defined

$$\mu = \frac{\kappa T}{\Delta^2}, \qquad \lambda = \frac{cT}{\Delta}, \qquad \eta = \frac{1}{1 + \sigma T}, \qquad \xi = \frac{b_1 T}{\Delta^2}.$$

The grid variable $w_{i,j}^n$ must be initialized at time steps $n = 0$ and $n = 1$; since, in the case of plate reverberation, the plate will be assumed to be at rest initially, these values may be set to zero. For a rectangular plate, the spatial domain of the grid function will be limited to $i \in [0, \dots, N_x]$, $j \in [0, \dots, N_y]$, where $N_x = \lfloor \frac{L_x}{\Delta}$, and $N_y = \lfloor \frac{L_x}{\Delta}$. As Δ is chosen to be the same in both the x and y directions, it will be true that for arbitrary L_x and L_y, there will be some error in this approximation. Because, for actual plate reverberation units, the number of grid points will be quite large, this effect will not have a large perceptual impact. Difference scheme (5) holds at interior points in the domain; near the edges, these must be modified, as will be discussed in Section 3.2.

3.1. Stability

For a linear time-invariant system such as the Kirchhoff plate, a stability bound may be simply obtained using von Neumann (or Fourier-based) methods [10, 7]. For scheme (5), the condition is

$$\Delta^2 \geq 2b_1T + c^2T^2 + \sqrt{(2b_1T + c^2T^2)^2 + 16\kappa^2T^2}. \qquad (6)$$

This condition, however, does not take into account the effects of boundary termination, which will discussed in Section 3.2; a more careful analysis must be carried out, perhaps making use of energy-based analysis techniques.

In general, the audio sample rate (and thus T) are fixed, and thus Δ may be chosen according to (6) above; it is advantageous to choose Δ as close to this bound as possible, in order to avoid excessive numerical dispersion [10].

3.2. Boundary Conditions

As mentioned above, the scheme (5) needs to specialized near the boundary. This can be done through discretization of boundary conditions (4). In the case of the free plate, the first of conditions (4) may be discretized as

$$w^n_{-1,j} = 2(1+\nu)w^n_{0,j} - w^n_{1,j} - \nu(w^n_{0,j+1} + w^n_{0,j-1}), \quad (7)$$

and the second of conditions (4) may be discretized as

$$w^n_{-2,j} = 2w^n_{-1,j} - 2w^n_{1,j} + w^n_{2,j} - (2-\nu)(w^n_{-1,j+1} - 2w^n_{-1,j} + w^n_{-1,j-1} - w^n_{1,j+1} + 2w^n_{1,j} - w^n_{1,j-1}). \quad (8)$$

3.3. Input/output

It is assumed here (and this is by no means essential) that the driving term is a point source, of the form $f(x,y,t) = g(t)\delta(x - x_{in}, y - y_{in})$; where, x_{in} and y_{in} are the coordinates of the source of strength $g(t)$ (i.e., $g(t)$ is the input waveform). (In a typical plate reverberation setup, x_{in} and y_{in} are generally fixed, but in simulation, they could well be dynamic—if the motion of the readin points is slow relative to audio rates, this leads to a very delicate phasing effect.) The output is read directly from the computed values of $w^n_{i,j}$ at each time step, at a point with coordinates x_{out} and y_{out}.

In general, the readin and readout coordinates will not lie directly on grid points; in the case of readout, simple bilinear interpolation using values at the four neighboring points suffices to determine, and in the case of input it is possible to "bilinearly" spread the input to the nearest four neighbors. More details are presented in [2]. It is worth noting that there is no limit to the number of input or output channels in such a simulation.

4. PLATE REVERBERATION

The main simplifications of this model with respect to the actual plate system, are the damping model already discussed, the free-free boundary conditions and the fact that constant tension is considered all over the plate.

4.1. Numerical Results

Table 1 presents the parameters used in simulations here: typical plate reverberation dimensions [3] and physical properties of steel (which is usually used as the plate material). In order to systematically test the reverberation

Length	$L_x = 2$ m
Width	$L_y = 1$ m
Thickness	$h = 0.0005$ m
Density	$\rho = 7860$ kg m^{-3}
Young modulus	$E = 2 \times 10^{11}$ N m^{-2}
Poisson's ratio	$\nu = 0.30$
Tension	$T = 680$ N/m $\Rightarrow c^2 = 173$ m^2 s^{-2}

Table 1. Parameters used in plate simulations.

characteristics of the model we simulated the response at one observation point of the plate subjected to a short-duration impulse loading applied at an excitation point. These simulations were carried out for various different sets of the damping parameters σ and b. In real plate reverberations the output is an acceleration signal. In our simulations, the acceleration is obtained from the model output displacement using the standard finite difference approximation of the second time differential operator [2].

For the analysis of room reverberation Schroeder [8] pointed out that the decay curve is obtained from a mesure of the impulse response $h(t)$ as

$$EDC_h(t) = \int_t^\infty h^2(\tau)d\tau. \quad (9)$$

$EDC_h(t)$ is the residual energy after time t on the impulse response. Simulated impulse responses have been band filtered by one-third-octave filters and the reverberation time T_{60} (the time required for a sound decaying of 60 dB) values have been obtained for each band. Figure 1 shows the behaviour of $EDC_h(t)$ for the frequency bands of a plate impulse response acceleration.

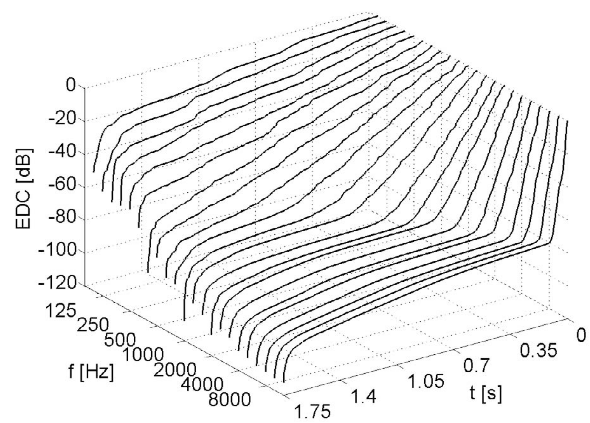

Fig. 1. Temporal evolution of $EDC_h(t)$ in third octave bands for $b = 0.002$ and $\sigma = 1$.

Figure 2 and Figure 3 indicate the influence of damping parameters σ and b on the reverberation times. As predicted by the damping model, σ is significant at low frequencies and b becomes increasingly at high frequencies.

4.2. Computational Complexity

The scheme (5) requires six multiplies and eight adds per grid point, per time step. For each time step, updating

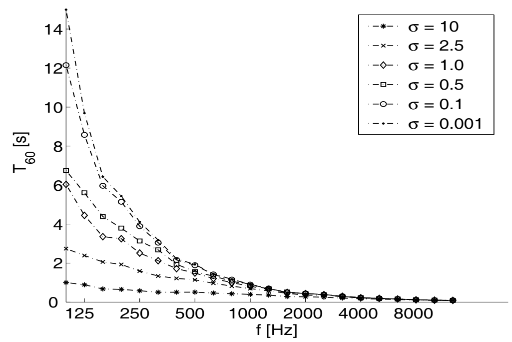

Fig. 2. T_{60} band-values for the plate acceleration impulse response; $b = 0.002$ fixed and various values of σ.

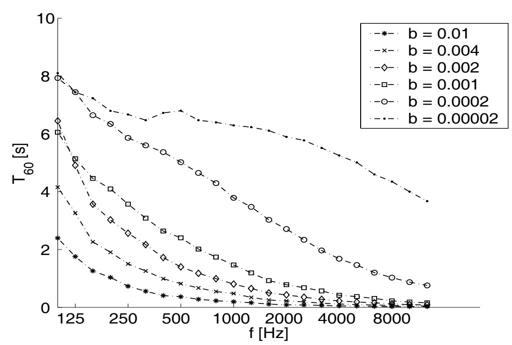

Fig. 3. T_{60} band-values for the plate acceleration impulse response; $\sigma = 1$ fixed and various values of b.

the entire grid will require roughly $14L_xL_y/\Delta^2$ operations, and thus the number of operations per second required will be $14L_xL_y/\Delta^2T$. Considering the simplest case of $c = b_1 = 0$, and assuming that (6) is satisfied with equality (which is nearly true in practice), we will then have $14L_xL_y/4\kappa T^2$ operations per second.

In the case of the plate discussed here at the sample rate $F_s = 44100Hz$ the order of operations per second is about $1.7 \cdot 10^{10}$, which is large, but not unmanageably so for a state of the art desktop computer.

5. CONCLUSIONS AND FUTURE DIRECTIONS

In this article, we have presented a straightforward numerical simulation routine, suitable for use in plate reverberation; such an algorithm extends the behaviour of a physical plate reverberation unit in that it becomes possible to vary the physical constants of the plate or its geometry (as well as input and output locations) in a very simple way.

We have only analyzed the influence of damping parameters on the reverberation time; it would be of great use to extend the analysis to other acoustical parameters such as clarity or center time and also to attempt to link plate parameters with typical room configurations by physical analysis or multidimensional perceptual analysis. At a more fundamental level, a better model for damping, both intrinsic and at the boundaries, is necessary for a complete simulation of a real reverberation unit.

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