dWFS: *DISTRIBUTED* WAVE FIELD SYNTHESIS

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ABSTRACT

In this paper we consider the problem of using a network of sensor and actuator nodes, to synthesize a given desired wave field in a given medium – we refer to this problem as the *distributed* Wave Field Synthesis problem (dubbed dWFS). We formulate this problem as one of minimization of a quadratic function, subject to linear (and very sparse) constraints. This formulation results from a finite-element approximation of the underlying wave equation, which constrains the values that a field and the source that induces that field can take. We present a complete solution for a 1D vibration, although our solution method can be readily extended to any number of dimensions. Numerical simulations are included.

1. INTRODUCTION

Many physical processes include transmission of wave patterns through various media. One of the simplest approaches in analyzing such processes is the use of wave equation in the linear form. Assuming homogeneity of material properties of the medium (i.e., density, elasticity) and that the pressure varies linearly with density inside its volume, a Newtonian formulation gives the linear wave equation [1]. An example is in acoustics where the linear wave equation is used to relate the air pressure variations with the forces that cause these variations.

In this work, we study the one dimensional wave equation using a finite element linear model in space-time. The general form of the one dimensional linear wave equation is given by:

$$\frac{\partial}{\partial x}[\alpha\frac{\partial\Phi}{\partial x}] - \frac{\partial}{\partial t}[\beta\frac{\partial\Phi}{\partial t}] = -F, \tag{1}$$

where α, β are constants depending on the properties of the medium, Φ is the wave field function and F is the source function. For the above case, the solution has been well studied in [1]. However, a difficult scenario arises when these parameters vary in space and time. For example, space variation could be an inhomogeneous material medium whose density varies as a function of space. A time variation, can be a moving object inside a medium where the material properties at a given spatial location are changing in time. The finite element method provide solutions to this class of problems and hence is the natural choice for analysis in this paper.

Various groups, primarily in Germany and in the Netherlands, have studied the WFS problem, under a different formulation based on Huygens' principle for wave propagation. WFS is a technique which has been studied using the Kirchoff integral in [3]. In the case of spatial and temporal inhomogeneities, using the traditional techniques mentioned in [3], WFS appears to be a complex problem, while using the proposed finite element method the solution is more straightforward. A number of references on varios aspects of the WFS problem are available from http://www.lnt.de/LMS/.

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2. A FINITE-ELEMENT APPROXIMATION OF THE WAVE EQUATION

2.1. Decomposition of the Space-Time Domain

In order to apply a space-time finite element method we decompose the domain Ω in which (1) is defined into square space-time elements. In the simple case of a continuous medium of length X space units and assuming a time interval of length T time units the defined domain $\Omega = [0, X] \times [0, T]$ can be decomposed in N square elements of side α ($\Delta\Omega^{(n)}_{n=1,\ldots,N}$) as Fig. 1 displays.

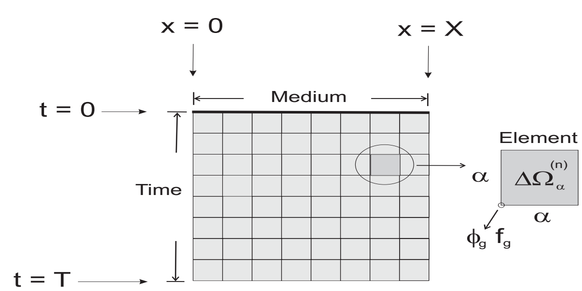

Fig. 1. Spatio-temporal decomposition of a 2D domain

This decomposition can be extended to problems in higher dimensions. Also for discontinuous media the previous technique is also valid. In the case of n spatial dimensions we use $d + 1$ dimensional hypercubes as elements. For example, to analyze $d = 2$ dimensional wave propagation phenomena we use $d + 1 = 3$ dimensional (i.e., cubic elements) and space-time surfaces.

2.2. Building a Linear Approximation

In the decomposition above, a set of G global space-time points lying on the apexes of the square elements is defined by the pairs $(x_g, t_g)_{g=1,\ldots,G}$. At these points the values of the field function $\Phi(x_g, t_g)$ and the source function $F(x_g, t_g)$ can be compactly written as a field vector $\{\Phi\} = \{\Phi(x_g, t_g)\}$ and a source vector $\{F\} = \{F(x_g, t_g)\}$ respectively. As shown in [8], there exists a linear relationship between the previous vectors expressed as:

$$[K_{G\times G}]\{\Phi\} + \{F\} = 0 \tag{2}$$

The entries of the matrix $[K_{G\times G}]$ depend on the geometry of the medium and the material properties of the medium captured by the parameters α_n and β_n which are the values of the parameters α and β appeared in (1) inside each elemental domain $\Delta\Omega_a^{(n)}$. Also α_n and β_n are assumed to be constant $\forall(x,y) \in \Delta\Omega^{(n)}$.

The entries K_{ij} of the matrix $[K_{G\times G}]$ can be determined by noting that if the pair (i,j) is not associated with any side of any element in the decomposition of Ω then the entry K_{ij} is zero, otherwise its value can be calculated as shown in [8]. Therefore, since in the decomposition of Ω not every space-time point is connected with every other space-time point, unless they belong to the same side of an element, we conclude that most of the entries of $[K_{G\times G}]$ are zero making the matrix sparse. Fact that facilitates many linear operations [6]. The system in (2) is a space-time finite element representation of (1).

The values of the field at the intermediate points inside the elemental domains were calculated by a simple linear interpolation. Such interpolation for one square elemental domain $\Delta\Omega^{(n)}$, is shown with the dashed quadruple in Fig. 2.

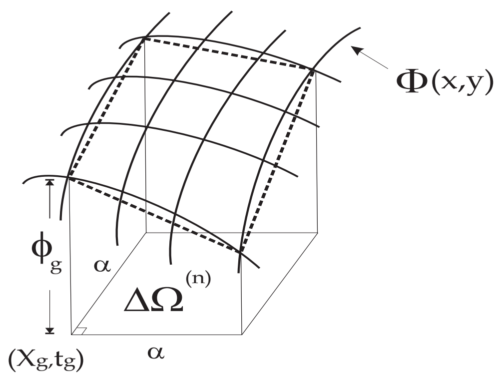

Fig. 2. Linear interpolation in a square elemental domain $\Delta\Omega^{(n)}$

3. APPLICATION TO FIELD SYNTHESIS

3.1. dWFS Problem Formulation

The problem of Wave Field Synthesis consists of the determination of a source function $F(x,t)$ such that, when used as an excitation driving the wave equation, the corresponding field of pressures that results approximates a given desired field $\Phi(x,t)$, to within an accepted degree of error [1, 2].

In a continuous two dimensional space-time domain Ω, we formulate the problem as follows. Given:

- A continuous domain $\Omega = [0,X] \times [0,T]$, with time/space varying properties determined by two functions $\alpha : \Omega \to \mathbb{R}$ and $\beta : \Omega \to \mathbb{R}$ (one for each dimension).
- A subset $\Omega_{\widetilde{F}} \subset \Omega$, as the support domain of the source function $\widetilde{F}$.
- A field function $\Phi \in L^2(\Omega)$.
- A differential operator that acts on the function Φ and is defined by:

$$\mathcal{L}[\Phi] = \frac{\partial}{\partial x}[\alpha\frac{\partial\Phi}{\partial x}] - \frac{\partial}{\partial t}[\beta\frac{\partial\Phi}{\partial t}] \tag{3}$$

This differential operator defines the wave equation mentioned in (1) namely:

$$\mathcal{L}[\Phi] + F = 0 \tag{4}$$

where F is defined as the source function. Also, based on the uniqueness of the solution of the wave equation in terms of Φ, we can define a new operator $\mathcal{L}'$, which given a source, maps it to the pressure field induced:

$$\mathcal{L}'[F] = \Phi. \tag{5}$$

Then, the WFS problem consists of solving a *constrained* inverse problem: we seek a source $\widetilde{F} : \Omega \to \mathbb{R}, \widetilde{F} \in L^2(\mathbb{R})$, that solves

$$\min_F ||\Phi - \mathcal{L}'[F]||_2,$$

subject to constraints on the locations on which it can take nonzero values:

$$\widetilde{F}(x,t) = 0,\ (x,t) \in \Omega - \Omega_{\widetilde{F}}.$$

By solving the above problem we will be able to give an approximate optimal solution in the "square error minimum sense" in a Wave Field Synthesis problem in the case of any imposed spatial and temporal constraint on the source function $\widetilde{F}$. Next we will show how the above problem can be treated as a problem of minimization of a quadratic function with linear constrains in a finite element setting.

3.2. Solution Using Linear Approximation

Consider a simple domain Ω, such as the one in Fig. 1, and its decomposition described in Section 2.1. Then one can divide the set of global space-time points $\{1,...,G\}$ of the decomposition into two subsets G_1 and G_2, in such a way that the subset G_1 contains all the global space-time points that belong to the support of the constrained source function $\widetilde{F}$ mentioned in Section 3.1, and G_2 contains the rest of the points. A possible division of the set of global space-time points of the domain of Fig. 1, is displayed in Fig. 3.

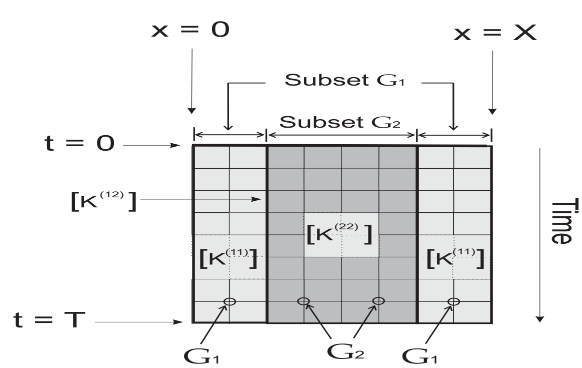

Fig. 3. Division of the set of global space-time points into two subsets G_1 and G_2

By expressing the values of the differences (errors) between the field to be synthesized and the synthesized field at the space-time points of the subsets G_1 and G_2 as vectors $\{e^{(1)}\}$ and $\{e^{(2)}\}$

respectively, one can write according to [8] the following system:

$$[K^{(11)}]\{e^{(1)}\} + [K^{(12)}]\{e^{(2)}\} + \{F^{(1)}\} = 0$$
$$[K^{(21)}]\{e^{(1)}\} + [K^{(22)}]\{e^{(2)}\} + \{F^{(2)}\} - \{\widetilde{F}\} = 0, \tag{6}$$

where $\{F^{(1)}\}$ and $\{F^{(2)}\}$ are vectors of values of the *direct* source function F at the space time points of the subsets G_1 and G_2 respectively. This direct source function F is associated with the field to be synthesized Φ via (1) in a continuous setting, or (2) in a finite element setting.

The vector $\{\widetilde{F}\}$ contains the values of the *constrained* field, at the points of the subset G_1 distributed source function that performs the dWFS.

The matrices $[K^{(kl)}], k, l \in \{1, 2\}$ are sub-matrices of the matrix $[K_{G\times G}]$. The entry $K_{ij}^{(kl)}$ $k, l \in \{1, 2\}$ $i \in G_k$ $j \in G_l$ is equal with the entry K_{ij} $i, j \in \{1, ..., G\}$ of the matrix $[K_{G\times G}]$.

Using the discretization above, the dWFS problem in the end consists of specifying the constrained source vector $\{\widetilde{F}\}$, such that the total square error:

$$E_{rr} = \{e^{(1)}\}^T\{e^{(1)}\} + \{e^{(2)}\}^T\{e^{(2)}\}$$

is minimized, and provided that the consumed energy does not exceed a given budget ($\{\widetilde{F}\}^T\{\widetilde{F}\} < E_{max}$).

According to [8], and by expressing the two error vectors $\{e^{(1)}\}$ and $\{e^{(2)}\}$ as $\{e^{(1)}\} = [A]\{\widetilde{F}\} + \{b\}$ and $\{e^{(2)}\} = [C]\{\widetilde{F}\} + \{d\}$, the sought solution can be given as:

$$\begin{aligned}
\{\widetilde{F}\} &= [[A]^T[A] + [C]^T[C]]^{-1}\{[A]^T\{b\} + [C]^T\{d\}\} \\
[A] &= [[K^{(11)}] - [K^{(12)}][K^{(22)}]^{-1}[K^{(21)}]]^{-1} \\
[C] &= -[K^{(22)}]^{-1}[K^{(21)}][A] \\
\{b\} &= [A]\{[K^{(12)}][K^{(22)}]^{-1}\{F^{(2)}\} - \{F^{(1)}\}\} \\
\{d\} &= [K^{(22)}]^{-1}\{\{F^{(2)}\} - [K^{(21)}]\{b\}\}
\end{aligned} \tag{7}$$

3.3. Quantification of the Error

Based on the previous derivations we conclude that two kinds of approximation errors are introduced:

- The linear interpolation of the field and the source function, inside the square elemental domains results in errors between these functions and their interpolated versions. Assuming continuity of these functions inside the domain Ω, fact that can be supported by physical considerations, the errors due to interpolation can be reduced by considering smaller sides of the square elemental domains (α).
- Another source of error is created by the difference between the original and synthesized wave field. A finite element version of this error written as $E = \{e_1\}^T\{e_1\} + \{e_2\}^T\{e_2\}$. This is the component of the error that we seek to minimize in our problem formulation.

3.4. Results

In the next figures the results of a simulation of a wave propagation in a line medium with 110 space points is presented. In Fig. 4, the sampled wave field is shown. The wave front was generated using an one dimensional finite difference model and a central piston source. Fig. 5 shows two snapshots of the resulting field, at two different points in time.

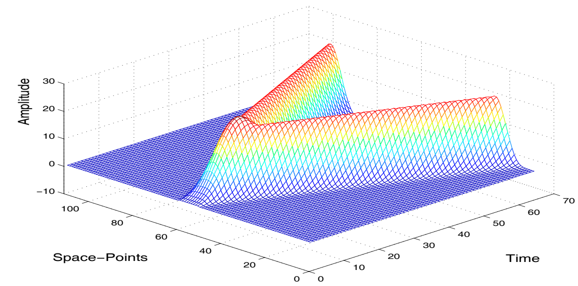

Fig. 4. Field generated by central excitation in 1D medium

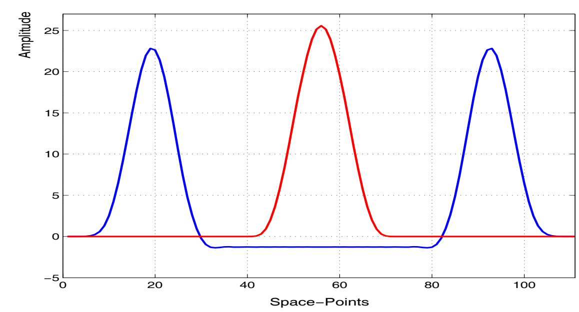

Fig. 5. Two snapshots of the propagated field

Next, Fig. 6 shows the source estimated to have produced the field observed in Fig. 4, based on samples of the observed wave field. The detection is based on the proposed finite element scheme and is obtained by directly solving for the vector $\{F\}$ equation (2) given the wave samples contained in $\{\Phi\}$.

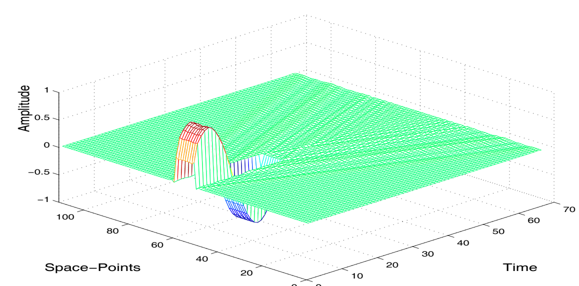

Fig. 6. Detected source based on the previous field data

Next, Fig. 7 shows the source obtained by computing (7), when the source is constrained to take non-zero values only in the region defined in Fig. 3. Finally, Fig. 8 shows the field generated by the constrained source. In this case, it is clear that after an initial transient behavior, the two fields become identical.

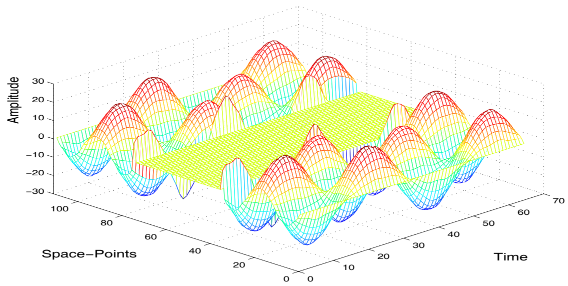

Fig. 7. Constrained Source generating the previous field with minimum error

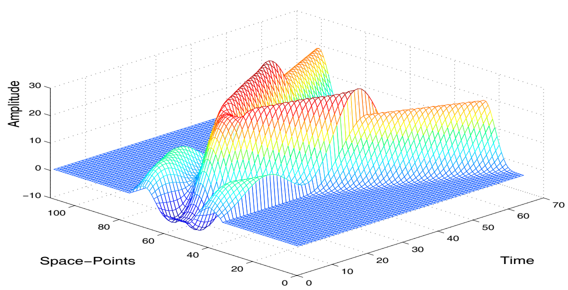

Fig. 8. Field generated by the constrained source.

4. CONCLUSIONS

4.1. Summary

In this paper, we have formulated and solved the problem of finding the best possible approximation to a given wave field, when generating this approximation using a distributed array of sources arbitrarily located in space.

We believe our formulation in terms of solving a quadratic minimization subject to linear constraints, besides other advantages that it may have over more traditional methods in terms of its ability to handle non-homogeneities, is particularly well suited for implementation on a distributed actuator array. This is because the matrix $K_{G\times G}$ is extremely sparse, and furthermore, its few non-zero entries are not uniformly spread over the whole domain Ω, but are contained within a small region. As a result, when the computation is distributed over a large number of actuator nodes, we expect that highly localized communication patterns will emerge – this is the main technical question we are investigating now.

4.2. On the Need to Increase the Resolution of WFS Systems

Why are we interested in the dWFS problem? So, far WFS has been studied under two important assumptions: a relatively small number of actuator nodes, and operated under the authority of a central controller. But, with large scale networks of sensors and actuator devices, equipped with wireless communication capabilities, and embedded in a physical medium, it should be possible to observe and control these signals with much higher resolution than is possible in the classical formulation of these problems.

We would like to point out that drastically increasing the resolution of a WFS system is not "an incremental tweak" on a classical problem, delivering more of the same – instead, both entirely new applications are enabled on the practical side, and the development of those applications requires new basic science, thus making it a worthy goal in itself. Consider, for example, the WFS problem in the context of acoustics. Spatial sound systems exist already, using about 5-7 speakers. But they require manual calibration, and their effectiveness is limited to a small area that is often referred to as the "sweet spot". If instead of using a few speakers we could use a few *hundred* of them, each one very small, and interconnected over a wireless network, the size of the sweet spot could be grown to cover a much larger area. And we believe that having the ability to shape the wave field *almost everywhere* (and not at isolated points in space) enables applications that are simply not possible with small sweet spots. For example, the creation of virtual environments, such as the meeting room of the Jedi council in the last Star Wars movie, as discussed in Fig. 9.

Fig. 9. A meeting of the Jedi council in "Star Wars III: Revenge of the Sith" – the holographic images correspond to council members attending the meeting from a remote location. If we are ever to realistically render a person moving freely around a room as part of a virtual reality system, it will be necessary to have the ability of placing a synthetic sound source at any arbitrary location in the room where we expect that person to move. This is possible with array technology (in this case a speaker array), but the number of speakers needed to enable a realistic rendering of a virtual acoustic source is many orders of magnitude higher than anything available today. We believe however that a very large number of micro-speakers, equipped with wireless communication capabilities, can do the trick.

Work in our lab, using an acoustic array comprising 64 speakers and 256 microphones, is currently under way with the goal of demonstrating a virtual mobile point source.

5. REFERENCES

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