SIMULTANEOUS DIAGONALIZATION WITH SIMILARITY TRANSFORMATION FOR NON-DEFECTIVE MATRICES

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ABSTRACT

The problem of joint eigenstructure estimation for the non-defective matrices is addressed. A procedure revealing the joint eigenstructure by simultaneous diagonalization with unitary and non-unitary similarity transformations alternately is proposed to overcome the convergence difficulties of previous methods based on simultaneous Schur form and unitary transformations. It can be proved that its asymptotic convergence rate is ultimately quadratic. Numerical experiments are conducted in a multi-dimensional harmonic retrieval application and suggest that the method presented here converges considerably faster than the methods based on only unitary transformation for matrices which are not near to normality.

1. INTRODUCTION

The problem of joint eigenstructure estimation for general non-defective matrices sharing the same set of eigenvectors is often encountered in many signal processing applications, e.g., 2-D DOA estimation [1], joint angle-delay estimation [2] and multidimensional harmonic retrieval [3].

In [1], the algebraically coupled matrix pencil (ACMP) method is proposed. It computes the Schur form of the first matrix whose Schur vectors are used in the trangularization of other matrices. This algorithm can't handle the case when the first matrix has repeated or more likely, very close eigenvalues. In [3], a Jacobi-type algorithm tries to find the simultaneous Schur form of the matrices to be estimated by orthogonal similarity transformations. This method is extended to the complex case in [4]. Similar with the one-matrix case [5] that may converge only linearly or does not converge at all, this type of scheme suffers the same convergence difficulty as will be shown in section 4. Another algorithm [6] tries to generalize the classical QR-algorithm to obtain simultaneous Schur form by the simultaneous QR-decomposition. However, unlike its one-matrix counterpart which is easily incorporated with acceleration strategies, this generalization losses essential properties for the one-matrix case and may result in a very slow convergence rate (if converge).

In this paper, by noting that the matrices encountered in the signal processing applications, whose joint eigenstructure is to be estimated, are non-defective, we turn to the strategy of simultaneously diagonalizing the matrices using both unitary and non-unitary transformations instead of the previous methods based on Schur form and unitary transformations. The non-unitary transformations are constrained to shear transformations which are used to bring the matrices closer to be normal ones. This strategy is closely related to [7, 8] for the non-simultaneous case of one matrix. Numerical experiments suggest that the method presented here converges considerably faster than the methods based on only unitary transformation for matrices which are not near to normality. And we can also prove that its asymptotic (for large SNR) convergence rate is ultimately quadratic while the proof is not included here due to page limitation.

2. PROBLEM FORMATION

Consider a set $\mathcal{A} = \{\boldsymbol{A}_n | n = 1, 2, \cdots, N\}$ of N complex or real $M \times M$ matrices. When the matrices in $\mathcal{A}$ are diagonalizable commuting matrices, then $\mathcal{A}$ can be simultaneously diagonalized. Hence, there is matrix $\boldsymbol{P}$ such that $\boldsymbol{A}_n = \boldsymbol{P}\boldsymbol{\Lambda}_n\boldsymbol{P}^{-1}$ for $n = 1, 2, \cdots, N$, where $\boldsymbol{\Lambda}_n$ is a diagonal matrix containing the eigenvalues of $\boldsymbol{A}_n$. $\boldsymbol{\Lambda}_n$ and their association, i.e., which eigenvalues correspond to the same simultaneous eigenvector are of our interest.

In practice, $\mathcal{A}$ is corrupted by estimation errors due to noise and finite sample size effects. Then the off-diagonal elements can only be minimized but cannot generally be driven to zero by similarity transformation. The *average eigenstructure* corresponds only to an approximate simultaneous diagonalization. To get insight to this *average eigenstructure*, taking the linear terms of the eigenvalue estimation error $\Delta\boldsymbol{\Lambda}_n$, see e.g.[2], we can express the estimation error of the ith eigenvalue of $\boldsymbol{A}_n$ as $\Delta\lambda_{n(i)} \approx q_i^H \Delta\boldsymbol{A}_n \cdot t_i$, where q_i and t_i are the simultaneous left and right eigenvectors of error-free $\mathcal{A}$ respectively. This equation shows that the dominant term of the estimation error $\Delta\lambda_{n(i)}$ is only related to q_i, t_i and $\Delta\boldsymbol{A}_n$ itself, but not related to the eigenvector error Δq_i and Δt_i. We are trying to minimize the off-diagonal norm by similarity transformation. But the approximation to the exact minimum solution is not critical to the performance, assuming that the simultaneous left and right eigenvectors suffer only small perturbations.

Hence, our task is: given noisy $\mathcal{A}$, find $\boldsymbol{P}$, such that the norm of the off-diagonal elements $\boldsymbol{P}^{-1}\mathcal{A}\boldsymbol{P}$ is minimized or approximately minimized. And it is preferred that the algorithm will drive the off-diagonal elements to zero in the noiseless case.

In this paper, we focus on the case $\mathcal{A}$ and its eigenvalues are both real. Then $\mathcal{A}$ can be simultaneously diagonalized with a real $\boldsymbol{P}$ and all the calculations are in the real domain. The extension to the complex case is not difficult [7, 9].

3. SHEAR-ROTATION ALGORITHM

The initial given matrices to be simultaneously diagonalized are denoted as $\mathcal{A}^{(0)} = \{\boldsymbol{A}_n^{(0)} | n = 1, \cdots, N\}$, we form the sequence $\mathcal{A}^{(k)}, k = 1, 2, \ldots$, by applying shear and unitary similarity transformation alternately,

$$\mathcal{A}^{(k+1)} = \boldsymbol{U}^{(k)H}\boldsymbol{S}^{(k)-1}\mathcal{A}^{(k)}\boldsymbol{S}^{(k)}\boldsymbol{U}^{(k)} \tag{1}$$

where $\boldsymbol{S}^{(k)}$ and the real unitary or orthogonal matrix $\boldsymbol{U}^{(k)}$ are identity matrix except for the elements

$$\begin{bmatrix} S_{pp}^{(k)} & S_{pq}^{(k)} \\ S_{qp}^{(k)} & S_{qq}^{(k)} \end{bmatrix} = \begin{bmatrix} \cosh y^{(k)} & \sinh y^{(k)} \\ \sinh y^{(k)} & \cosh y^{(k)} \end{bmatrix} \tag{2}$$

$$\begin{bmatrix} U_{pp}^{(k)} & U_{pq}^{(k)} \\ U_{qp}^{(k)} & U_{qq}^{(k)} \end{bmatrix} = \begin{bmatrix} \cos\theta^{(k)} & \sin\theta^{(k)} \\ -\sin\theta^{(k)} & \cos\theta^{(k)} \end{bmatrix} \tag{3}$$

It is easily to verify that the elements of $\boldsymbol{S}^{(k)-1}$ and $\boldsymbol{S}^{(k)}$ are equal except for the elements at (p, q) and (q, p), where p, q with $p < q$ are determined by k according to the choice of pivots order as discussed at the end of this section

$$\begin{aligned} [\boldsymbol{S}^{(k)-1}]_{pq} &= -S_{pq}^{(k)} = -\sinh y^{(k)} \\ [\boldsymbol{S}^{(k)-1}]_{qp} &= -S_{qp}^{(k)} = -\sinh y^{(k)} \end{aligned} \tag{4}$$

Each transformation affects only the pth and qth row and column of $\mathcal{A}^{(k)}$. And we put

$$\begin{aligned} \mathcal{A}'^{(k)} &= \boldsymbol{S}^{(k)-1}\mathcal{A}^{(k)}\boldsymbol{S}^{(k)} \\ \mathcal{A}''^{(k)} &= \boldsymbol{U}^{(k)T}\mathcal{A}'^{(k)}\boldsymbol{U}^{(k)} = \mathcal{A}^{(k+1)} \end{aligned}$$

In the remainder of this section, we will focus on the kth iteration and omit the superscript (k). After the shear transformation, the elements of the nth matrix of $\mathcal{A}'$, i.e. $\boldsymbol{A}'_n$, are

$$\begin{aligned} a'_{n,pj} &= \cosh y \cdot a_{n,pj} - \sinh y \cdot a_{n,qj} \\ a'_{n,qj} &= -\sinh y \cdot a_{n,pj} + \cosh y \cdot a_{n,qj} \\ a'_{n,ip} &= \cosh y \cdot a_{n,ip} + \sinh y \cdot a_{n,iq} \\ a'_{n,iq} &= \sinh y \cdot a_{n,ip} + \cosh y \cdot a_{n,iq} \\ a'_{n,pq} &= a_{n,pq} + W_n, \quad a'_{n,qp} = a_{n,qp} - W_n \\ a'_{n,pp} &= a_{n,pp} + V_n, \quad a'_{n,qq} = a_{n,qq} - V_n \end{aligned} \tag{5}$$

where $i, j \neq p, q$, and

$$\begin{aligned} V_n &= \sinh^2 y \cdot d_n + \frac{1}{2}\sinh 2y \cdot \xi_n \\ W_n &= \frac{1}{2}\sinh 2y \cdot d_n + \sinh^2 y \cdot \xi_n \\ d_n &= a_{n,pp} - a_{n,qq}, \quad \xi_n = a_{n,pq} - a_{n,qp} \end{aligned} \tag{6}$$

by the unitary transformation, we have

$$\begin{aligned} a''_{n,pj} &= \cos\theta \cdot a'_{n,pj} - \sin\theta \cdot a'_{n,qj} \\ a''_{n,qj} &= \sin\theta \cdot a'_{n,pj} + \cos\theta \cdot a'_{n,qj} \\ a''_{n,ip} &= \cos\theta \cdot a'_{n,ip} - \sin\theta \cdot a'_{n,iq} \\ a''_{n,iq} &= \sin\theta \cdot a'_{n,ip} + \cos\theta \cdot a'_{n,iq} \\ a''_{n,pq} &= a'_{n,pq} + Q_n, \quad a''^{(k)}_{n,qp} = a'_{n,qp} + Q_n \\ a''_{n,pp} &= a'_{n,pp} + P_n, \quad a''_{n,qq} = a'_{n,qq} - P_n \end{aligned} \tag{7}$$

again $i, j \neq p, q$, and

$$\begin{aligned} P_n &= -\sin^2\theta \cdot d'_n + \frac{1}{2}\sin 2\theta \cdot \xi'_n \\ Q_n &= \frac{1}{2}\sin 2\theta \cdot d'_n + \sin^2\theta \cdot \xi'_n \\ d'_n &= a'_{n,pp} - a'_{n,qq}, \quad \xi'_n = -a'_{n,qp} - a'_{n,pq} \end{aligned} \tag{8}$$

In order to show how to choose y and θ. We first denote $\boldsymbol{A}_n$ as the sum of a diagonal matrix $\boldsymbol{D}_n$ and a nondiagonal matrix $\boldsymbol{E}_n$ with zero diagonal

$$\boldsymbol{A}_n = \boldsymbol{D}_n + \boldsymbol{E}_n, \quad diag\{\boldsymbol{E}_n\} = \boldsymbol{0} \tag{9}$$

and we put

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_1 & \boldsymbol{E}_2 & \cdots & \boldsymbol{E}_N \end{bmatrix} \tag{10}$$

For the one matrix case where $N = 1$, at each step, y and θ are chosen to minimize (or approximately) the Frobenius norm $\|\boldsymbol{A}'_1\|_F$ and $\|\boldsymbol{E}''_1\|_F$ respectively [7] and it is proven that the ultimate convergence rate is quadratic. The underlying principle of this algorithm is: using shear transformation to transform $\boldsymbol{A}_1$ into a matrix $\boldsymbol{A}'_1$ with a smaller Frobenius norm in order to reduce the departure from normality (see [7], p.211 for details), while the Givens rotation tries to diagonalize $\boldsymbol{A}'_1$ into $\boldsymbol{A}''_1$ by minimizing the target off-diagonal norm $\|\boldsymbol{E}''_1\|_F$ with $\|\boldsymbol{A}''_1\|_F$ equal to $\|\boldsymbol{A}'_1\|_F$.

When $N \geq 2$, We hope to preserve the property that the asymptotic convergence rate is ultimately quadratic. To achieve this, while the proof is not given here due to page limitation, we first find h, such that

$$|a_{h,pp} - a_{h,qq}| = \max_{1 \leq n \leq N} |a_{n,pp} - a_{n,qq}| \tag{11}$$

and use $\boldsymbol{A}'_h$ to determine y, i.e., find y to (approximately) minimize $\|\boldsymbol{A}'_h\|_F$. While for θ,

$$\|\boldsymbol{E}''\|_F^2 = \sum_{n=1}^{N} \|\boldsymbol{E}''_n\|_F^2 \tag{12}$$

is used as the cost function to be minimized.

Now we discuss the choice of y and θ in detail. For the shear transformation, after choosing h, we follow [8],

$$\begin{aligned}\|\boldsymbol{A}'_h\|_F^2 &= (\cosh 2y - 1)G_h - 2\sinh 2y \cdot K_h \\ &+ \sinh^2 2y \cdot (d_h^2 + \xi_h^2) + \sinh 4y \cdot \xi_h d_h + \|\boldsymbol{A}_h\|_F^2\end{aligned} \tag{13}$$

where

$$G_h = \sum_{j \neq p,q} \left[a_{h,pj}^2 + a_{h,qj}^2 + a_{h,jp}^2 + a_{h,jp}^2\right] \tag{14}$$

$$K_h = \sum_{j \neq p,q} \left[a_{h,pj}a_{h,qj} - a_{h,jp}a_{h,jq}\right] \tag{15}$$

differentiating (13) with respect to y

$$\begin{aligned}\frac{\partial}{\partial y}\|\boldsymbol{A}'_h\|_F^2 &= 2\sinh 2y \cdot G_h - 4\cosh 2y \cdot K_h \\ &+ 2\sinh 4y \cdot (d_h^2 + \xi_h^2) + 4\cosh 4y \cdot \xi_h d_h\end{aligned} \tag{16}$$

take a linear approximation to the zeros of (16), i.e.,

$$\begin{aligned}\sinh 4y \approx 2\sinh 2y \approx 4\sinh y \\ \cosh 4y \approx \cosh 2y \approx \cosh y\end{aligned} \tag{17}$$

we get,

$$\tanh y = \frac{K_h - \xi_h d_h}{2(d_h^2 + \xi_h^2) + G_h} \tag{18}$$

For the unitary transformation, we have

$$\begin{aligned}&\|\boldsymbol{E}''\|_F^2 = \|\boldsymbol{E}'\|_F^2 + \\ &\sum_{n=1}^{N}\left[\frac{1}{2}\sin^2 2\theta \cdot (d_n'^2 - \xi_n'^2) - \frac{1}{2}\sin 4\theta \cdot \xi_n' d_n'\right]\end{aligned} \tag{19}$$

the first and second derivative of (19) with respect to θ are

$$\frac{\partial}{\partial \theta}\|\boldsymbol{E}''\|_F^2 = \sum_{n=1}^{N}\left[\sin 4\theta \cdot (d_n'^2 - \xi_n'^2) - 2\cos 4\theta \cdot \xi_n' d_n'\right] \tag{20}$$

$$\frac{\partial^2}{\partial \theta^2}\|\boldsymbol{E}''\|_F^2 = \sum_{n=1}^{N}\left[4\cos 4\theta \cdot (d_n'^2 - \xi_n'^2) + 8\sin 4\theta \cdot \xi_n' d_n'\right] \tag{21}$$

The minimum of (19) is obtained when the first derivative equals zero and the second derivative is greater than zero

$$\tan 4\theta = \frac{2\sum_{n=1}^{N}(\xi_n' d_n')}{\sum_{n=1}^{N}(d_n'^2 - \xi_n'^2)} \tag{22}$$

$$\cos 4\theta \cdot \sum_{n=1}^{N}(d_n'^2 - \xi_n'^2) + 2\sin 4\theta \cdot \sum_{n=1}^{N}\xi_n' d_n' > 0 \tag{23}$$

Similar with lemma 2.1 of [10], by the observation that (19) is $\pi/2$ periodic in θ, we can constrain $\theta \in [-\pi/4, \pi/4)$. It is easily shown that the two solutions of (22) has opposite value of (21), the one makes (21) positive, thus satisfies (23) is our expected solution.

It is easy to combine the shear and unitary transformation into one transformation matrix as [7]. Thus the additional computation due to the unitary transformation mainly occurs at the calculation of θ which only involves limited elements.

The pivots (p, q) are chosen cyclically by rows, i.e., in the order $(1,2)(1,3)...(1,N),(2,3)...(2,N),...,(N-1,N)$. And one cycle constitutes a *sweep*.

4. SIMULATION

We conduct our simulation in the scenario of multidimensional harmonic retrieval application based on multidimensional ESPRIT algorithm. The data model of [3] (p.162, eq.1) is applied where the signal s_i is assumed to be white Gaussian process and uncorrelated with each other. A 2-D uniform rectangular array (URA) with 6×6 elements is used. Four harmonic components are set as: $\boldsymbol{\mu}_1 = \pi[0.20, 0.23]^T$, $\boldsymbol{\mu}_2 = \pi[0.22, 0.20]^T$, $\boldsymbol{\mu}_3 = \pi[0.24, 0.26]^T$, $\boldsymbol{\mu}_4 = \pi[0.26, 0.23]^T$. The number of snapshots is 512. SNR is defined as per source per element. 2000 trials are conducted. Signal subspace estimation follows the real processing of [3]. To avoid frequency warping due to Cayley transformation, invariance equation is solved in the complex domain (except T algorithm [11]) by transforming the estimated real signal subspace back to complex domain. In the joint eigenstructure estimation procedure, we will face two complex matrices which can be diagonalized with a real matrix $\boldsymbol{P}$. Thus, the problem is equivalent to simultaneously diagonalizing four real matrices with a real $\boldsymbol{P}$.

The plots show the RMSE (with $\boldsymbol{\mu}$ normalized by π) vs. SNR for the harmonic estimates of various methods at different iteration counts. The resulting RMS errors of dimension 1 and 2 are depicted in Fig. 1 and 2 respectively. It is shown that the shear-rotation algorithm converges to reliable estimates within 16 iterations while the SSD [3] and simultaneous QR algorithm still have large RMS errors even at 100 and 500 iterations respectively. Unlike the one-dimensional case which the triangularization and diagonalization based method reach the same estimates if $\boldsymbol{P}$ is not constrained to be real, the ultimate performance of SSD with exhausted iterations (if converge) may be different with shear-rotation algorithm. The performance difference depends on each particular parameter and both algorithms have better and worse estimates than the other. Shear-rotation algorithm is consistent with the T algorithm [11] when the latter works (T algorithm may fail when $\boldsymbol{\mu}$ is relatively close to $\pm\pi$ and cannot be extended to cases more than two dimensions), thus allowed a closed form performance analysis [2][12] while SSD eludes such an analysis.

Numerous experiments are conducted while only one case is demonstrated here. It is shown that the convergence difficulty is severe for methods based on unitary transformation when the harmonic components are relatively close. This corresponds to a larger departure from normality of the matrices to be estimated. In a more extreme case, 20 harmonic components are equally spaced as $\boldsymbol{\mu}_1 = \pi[0, 0]^T$, $\boldsymbol{\mu}_2 = \pi[0.02, 0.02]^T$, $\cdots$, $\boldsymbol{\mu}_{20} = \pi[0.38, 0.38]^T$. For 20×20 URA, 256 snapshots and infinite SNR, the shear-rotation algorithm will converge at around 80 iterations. While for SSD convergence is not observed even at 1 million iterations leaving the harmonics un-resolved .

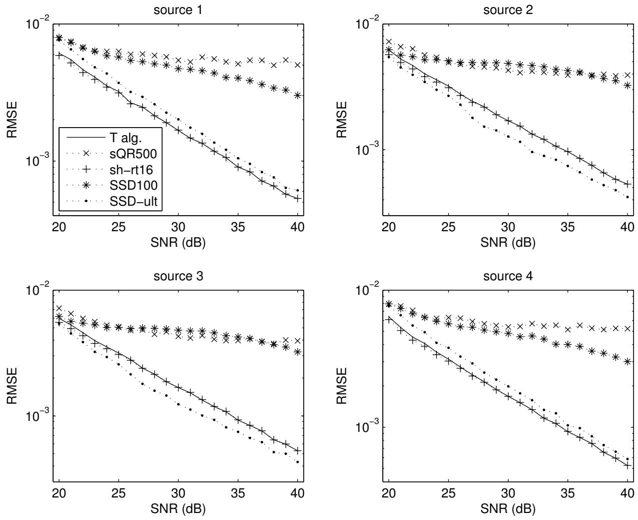

Fig. 1. Performance comparison for dimension 1. (T alg.: *T* algorithm; sim-QR-500: *simultaneous QR* at 500 iterations; sh-rt16: *shear-rotation* at 16 iterations; SSD-100: *SSD* at 100 iterations; SSD-ult: ultimate performance of *SSD*)

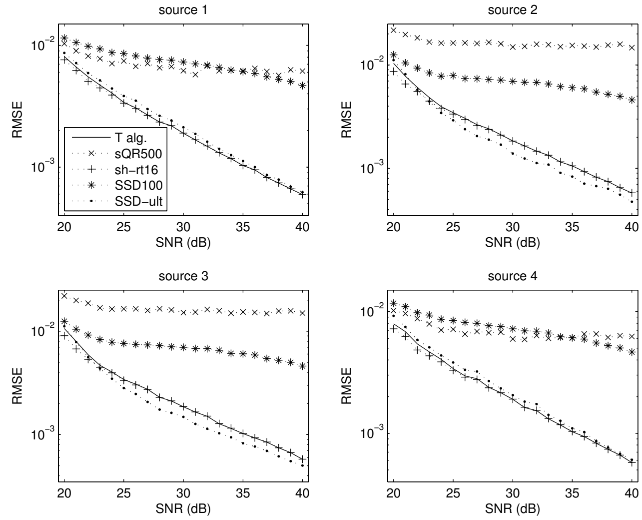

Fig. 2. Performance comparison for dimension 2.

5. CONCLUSION

We have proposed a simultaneous diagonalization algorithm for non-defective matrices sharing same set of eigenvectors to overcome the convergence difficulties of previous joint eigenstructure estimation methods based on simultaneous Schur form and unitary transformations. The advantage of this algorithm is its good convergence property while computations per iteration is comparable with unitary transformation based methods. And the performance is consistent with theoretical analysis. It can be proved that its asymptotic convergence rate is ultimately quadratic. The global convergence is convinced empirically, but we have not been able to give a proof. Numerical experiments in a multi-dimensional harmonic retrieval application validate that the method presented here converges considerably faster than the methods based on only unitary transformation for matrices which are not near to normality.

6. ACKNOWLEDGEMENT

This work was supported by National Natural Science Foundation of China under Grants 60572072 and 60496311, the China High-Tech 863-FuTURE Project under Grant 2003AA123310, and the International Cooperation Project on Beyond 3G Mobile of China under Grant 2005DFA10360.

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