ON A STOCHASTIC VERSION OF THE PLENACOUSTIC FUNCTION

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ABSTRACT

In this paper, we model a spatially varying channel where a source is moving along a random trajectory with respect to a fixed receiver. The aim is to compute the power spectral density corresponding to the channel impulse response as a function of temporal and spatial frequencies. The trajectory of the source follows an autoregressive model where the poles of the system control the smoothness of the path. Theoretical results are presented for the AR2 case and generalized to any ARn systems. Simulations results are shown and compared to the presented theory. The stochastic plenacoustic function analyzed in this paper also models time-varying channels governed by the wave equation, like they appear in acoustic echo cancellation or ultra wide band communication channels.

1. INTRODUCTION

In this paper, we introduce a stochastic version of the plenacoustic function (PAF). The PAF has been introduced in [1] and studied in more details in [2]. The PAF characterizes the sound field in space, eg. inside a room. It contains all the information about the spatial evolution of the sound field. The PAF was first studied along a line in a room. There it was observed that the two dimensional Fourier transform of the PAF has a bow-tie shape. Studying this specific shape led to interesting results regarding sampling and interpolation of the sound field. It allowed to state how many microphones (or loudspeakers) are necessary to sample and reconstruct the sound field up to a certain temporal frequency and with a specific SNR on the reconstruction. The study of the PAF has been further generalized to planar or circular arrays of microphones [2, 3].

In this paper, we consider a source v placed at an arbitrary position on a curve $\Gamma(s)$ and a receiver r at a constant position as shown in Fig. 1. $\Gamma(s)$ is a random process describing the trajectory and s is its independent variable. s can be seen as a curved abscissa and therefore s corresponds to the position of the source. We are interested in modeling the channel between the source and the receiver. We therefore imagine that at each position $\Gamma(s)$ the source sends a Dirac pulse and the impulse response is measured at the receiver. The measured room impulse response is a function of the time t representing the duration of the recorded impulse response. Considering all the room impulse responses measured from each position along the random trajectory, the total dataset recorded by the microphone is a function of both s and t. We call this dataset the stochastic plenacoustic function (SPAF) $p(s,t)$[1]. In order to characterize the channel from the source to the microphone, this paper studies the autocorrelation and the power spectral density (PSD) of the SPAF.

The study presented in this paper finds applications in the modeling of channel impulse responses for acoustic and electromagnetic waves. A model for this kind of channel impulse responses could be used in echo cancellation problems where one would adapt the rate of changes of the adaptive algorithms per temporal frequency band. In the electromagnetic case, a possible application is the communication between a fixed and a mobile station where the equalizer has to be adapted for multipath cancellation. As will be observed in this paper, the PSD of our process has a narrow support along its spatial frequency at low temporal frequencies. The rate of change could then be lower than for higher temporal frequencies where the spatial support is large. This frequency dependent choice of the rate of change in adaptive algorithms is a direct application of our work and is matter of current research.

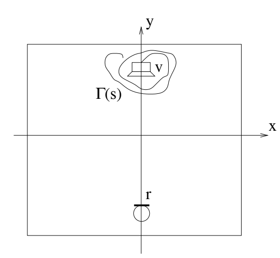

Fig. 1. Setup of the problem.

2. MODEL FOR THE MOVING SOURCE

The typical movement of an object is described by differential equations. For simplicity, we consider a linear model described by a state-space system in controller canonical form:

$$\begin{cases} \dot{\mathbf{U}}(s) = F\mathbf{U}(s) + \mathbf{b}W(s) \\ Y(s) = \mathbf{c}^T\mathbf{U}(s), \end{cases} \tag{1}$$

[1] In this paper, we present s as a parameter corresponding to the position along the trajectory. An other alternative is to consider the time-varying source along the trajectory. In that case, the time-varying channel would be represented by the channel impulse response $h(s,t)$ where s would be the delay of the emitted pulse and t the time. It can then be easily proven that $h(s,t) = p(s, t-s)$.

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with

$$F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}.$$

$\mathbf{U}(s)$ corresponds to the state variables of the system and $W(s)$ corresponds to the input signal. The output of the system is $Y(s)$. In our study, we consider the input $W(s)$ of the system to be a white Gaussian stationary process. Further, the output of the system is the state variable corresponding to the position. With this choice, it can be shown that the output of the system $Y(s)$ follows precisely a continuous autoregressive (AR) model. The order of the model is dependent on the size of the matrix F in (1). For simplicity, we will present the study of the autocorrelation function of a source moving following an AR2 law. Further, the results will be generalized for any AR order. As will be discussed further, increasing the order of the autoregressive model allows to create a smoother output of the system.

2.1. AR2 Process

In the two dimensional case, the two state variables are position $X(s)$ and velocity $V(s)$. The input $W(s)$ is a white Gaussian stationary process. To write the system (1) in two dimensions, we set

$$\mathbf{U}(s) = \begin{bmatrix} X(s) \\ V(s) \end{bmatrix},\ F = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ \mathbf{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (2)$$

Using (2), (1) can be rewritten as

$$\begin{cases} \ddot{X}(s) = -a_0 X(s) - a_1 \dot{X}(s) + W(s) \\ Y(s) = X(s). \end{cases} \quad (3)$$

The correlation of $Y(s)$ is defined as

$$R_Y(s) = E[Y(l+s)Y(l)]. \quad (4)$$

Since $Y(s)$ is a stationary process, $R_Y(s)$ does not depend on l.

If the system is stable, only the steady state term is present at the output, i.e.

$$Y(s) = \int_{-\infty}^{s} \mathbf{c}^T e^{F(s-l)} \mathbf{b} W(l) \mathrm{d}l. \quad (5)$$

Therefore, $R_Y(s)$ for $s \geq 0$ can be written as follows:

$$\begin{aligned} E\left[\int_{-\infty}^{s}\int_{-\infty}^{0} \mathbf{c}^T e^{F(s-l_1)} \mathbf{b} W(l_1) W(l_2) \mathbf{b}^T e^{F^T(-l_2)} \mathbf{c} \mathrm{d}l_1 \mathrm{d}l_2\right] \\ = \sigma^2 \int_{-\infty}^{0} \mathbf{c}^T e^{F(s-l)} \mathbf{b}\mathbf{b}^T e^{F^T(-l)} \mathbf{c} \mathrm{d}l, \end{aligned} \quad (6)$$

using that $E[W(l_1)W(l_2)] = \sigma^2 \delta(l_1 - l_2)$. For simplicity, we assume that the matrix F can be diagonalized. If this is not the case, similar results can be derived by using the Jordan decomposition [4]. Then, F can be decomposed using its eigenvalues λ_1 and λ_2 and eigenvector matrix V. Therefore,

$$e^{Fs} = V \begin{bmatrix} e^{\lambda_1 s} & 0 \\ 0 & e^{\lambda_2 s} \end{bmatrix} V^{-1} = \begin{bmatrix} L(s) & M(s) \\ N(s) & O(s) \end{bmatrix}. \quad (7)$$

Using (7), it can be shown that (6) can be rewritten as

$$R_Y(s) = \sigma^2 \int_{-\infty}^{s} M(s-l) M(-l) \mathrm{d}l, \quad (8)$$

with $M(s-l) = C_1 e^{\lambda_1(s-l)} + C_2 e^{\lambda_2(s-l)}$ where C_1 and C_2 are obtained as $C_1 = V[1,1]V^{-1}[1,2]$ and $C_2 = V[1,2]V^{-1}[2,2]$.

By integrating (8), and taking into account that $R_Y(s)$ is an even function, we obtain

$$\begin{aligned} R_Y(s) = -\sigma^2 \Bigg(e^{\lambda_1 |s|} \left[\frac{C_1^2}{2\lambda_1} + \frac{C_1 C_2}{\lambda_1 + \lambda_2}\right] + \\ e^{\lambda_2 |s|} \left[\frac{C_2^2}{2\lambda_2} + \frac{C_1 C_2}{\lambda_1 + \lambda_2}\right] \Bigg). \end{aligned} \quad (9)$$

Introducing now the random vector $\mathbf{Y} = [Y(0)Y(s)]^T$, the correlation matrix $R_{\mathbf{Y}}$ is

$$R_{\mathbf{Y}} = \begin{bmatrix} E[Y(s)Y(s)] & E[Y(s)Y(0)] \\ E[Y(0)Y(s)] & E[Y(s)Y(s)] \end{bmatrix}, \quad (10)$$

the probability density function of $\mathbf{Y}$ is given by:

$$f_{\mathbf{Y}}(\mathbf{y}) = \frac{1}{2\pi\sqrt{|R_{\mathbf{Y}}|}} e^{\frac{-R_Y(0)}{2|R_{\mathbf{Y}}|}\left(y_0^2 + y_s^2 - \frac{2R_Y(s)}{R_Y(0)} y_s y_0\right)}. \quad (11)$$

Note that in this expression, y_s corresponds to a realization of the random variable $Y(s)$ and y_0 to a realization of the random variable $Y(0)$.

2.2. Generalization to ARN process

In the case of an ARN process the previous formulae can be generalized. The matrix F is still considered to be diagonalized and therefore can be decomposed in its N eigenvalues $\lambda_1 \dots \lambda_N$ and N eigenvectors represented by the eigenvector matrix V of dimension N. The autocorrelation of the process $Y(s)$ is then

$$R_Y(s) = -\sigma^2 \left[\sum_{i=1}^{N} \left(\sum_{j=1}^{N} \frac{C_i C_j}{\lambda_i + \lambda_j}\right) e^{\lambda_i |s|}\right], \quad (12)$$

where $C_i = V[1,i]V^{-1}[i,N]$.

In order to compare different ARN processes, one has to normalize the noise input signals. The normalization is chosen to obtain a unitary variance for the output of the system, $Y(s)$. For this, one simply needs to impose that

$$\sigma^2 = -\frac{1}{\sum_{i=1}^{N}\left(\sum_{j=1}^{N} \frac{C_i C_j}{\lambda_i + \lambda_j}\right)}. \quad (13)$$

3. STOCHASTIC PLENACOUSTIC FUNCTION

In this section, the stochastic plenacoustic function is introduced. Further, the channel corresponding to a source on a random trajectory with respect to a fixed receiver is analyzed in time and frequency domain. In order to characterize the channel impulse response $p(s,t)$, we send a signal $v(s,t)$ and measure $r(t)$, the receiver signal. $v(s,t)$ represents a two dimensional signal. It is function of s the position and t the time. In the following, s is considered as an independent variable not related to t. We therefore have that

$$r(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v(s,u) p(s,t-u) \mathrm{d}u \mathrm{d}s. \quad (14)$$

Replacing $v(s,t) = \delta(t)\delta(s - s_1)$, the output is $r(t) = p(s_1, t)$.

For simplicity of the calculations, a few assumptions will be added to the model. First, the movement of the source will follow an AR model along one line as shown in Fig. 2. We consider a mean distance between source and receiver of d.

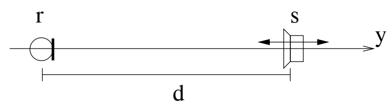


Fig. 2. Setup of the problem.

For each possible position of the source we consider the room impulse response between that position and the receiver. In the further mathematical derivations only the free field situation will be considered. In that case, one can rewrite the SPAF as in [2]

$$P(s,t) = \frac{\delta(t - \frac{|Y(s)+d|}{c})}{4\pi|Y(s)+d|}, \qquad (15)$$

with c the speed of sound propagation.

Another assumption done to simplify the calculations, is to consider the receiver far enough from the source to neglect the attenuation depending on the distance, i.e. the denominator in (15). Also, we consider d to be at least a few times larger than the standard deviation of the process Y. With those assumptions, we consider the Fourier transform of the SPAF along the temporal frequency axis and call this process $Z(s,\omega)$ defined as

$$Z(s,\omega) = e^{-j\frac{\omega}{c}(d+Y(s))}. \qquad (16)$$

The covariance function of the process $Z(s,\omega)$ can be calculated as follows for $s \geq 0$:

$$\begin{aligned} R_Z(s,\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\frac{\omega}{c}(y_s - y_0)} f_{\mathbf{Y}}(\mathbf{y})\mathrm{d}y_s\mathrm{d}y_0 - E^2[Z] \quad (17) \\ &= C\int_{-\infty}^{\infty} e^{-j\frac{\omega}{c}y_0\left(\frac{R_Y(s)}{R_Y(0)}-1\right)} e^{\frac{R_Y(0)}{2|R_{\mathbf{Y}}|}y_0^2\left(\frac{R_Y^2(s)}{R_Y^2(0)}-1\right)}\mathrm{d}y_0 \\ &\quad \int_{-\infty}^{\infty} e^{-j\frac{\omega}{c}y_s} e^{-\frac{R_Y(0)y_s^2}{2|R_{\mathbf{Y}}|}}\mathrm{d}y_s - E^2[Z], \quad (18) \end{aligned}$$

with $C = \frac{1}{2\pi\sqrt{|R_{\mathbf{Y}}|}}$. Using $E[e^{j\omega Y}] = e^{jE[Y]\omega - \frac{Var[Y]\omega^2}{2}}$ and [5], (18) becomes in the AR2 case

$$\begin{aligned} R_Z(s,\omega) &= e^{-\left(\frac{\omega}{c}\right)^2(R_Y(0)-R_Y(s))} - e^{-\left(\frac{\omega}{c}\right)^2 R_Y(0)} \quad (19) \\ &= e^{-\left(\frac{\omega}{c}\right)^2\left[D_1\left(1-e^{\lambda_1|s|}\right)+D_2\left(1-e^{\lambda_2|s|}\right)\right]} \\ &\quad -e^{-\left(\frac{\omega}{c}\right)^2(D_1+D_2)}, \quad (20) \end{aligned}$$

with $D_1 = -\sigma^2\left(\frac{C_1^2}{2\lambda_1} + \frac{C_1C_2}{\lambda_1+\lambda_2}\right)$ and $D_2 = -\sigma^2\left(\frac{C_2^2}{2\lambda_2} + \frac{C_1C_2}{\lambda_1+\lambda_2}\right)$.

In the generalized case of an ARN process, it can be shown that

$$R_Z(s,\omega) = e^{-\left(\frac{\omega}{c}\right)^2\left[\sum_{i=1}^N D_i\left(1-e^{\lambda_i|s|}\right)\right]} - e^{-\left(\frac{\omega}{c}\right)^2\sum_{i=1}^N D_i}, \quad (21)$$

with $D_i = -\sigma^2\sum_{j=1}^N \frac{C_iC_j}{\lambda_i+\lambda_j}$.

3.1. Power spectral density of the SPAF

The PSD $S_Z(\gamma,\omega)$ is given by the Fourier transform of the correlation function, i.e.

$$S_Z(\gamma,\omega) = \int_{-\infty}^{\infty} e^{-j\gamma s} R_Z(s,\omega)\mathrm{d}s. \qquad (22)$$

We are currently investigating a closed-form solution of such an integral. Nevertheless, numerical evaluation of (21) and of its Fourier transform are shown in Fig. 3(a) and Fig. 3(b) for an AR2 case. As can be seen from Fig. 3(b), the PSD of the SPAF has most of its energy contained in a bow-tie region. The intuition behind this result is the following. Considering a source emitting a waveform of very low temporal frequency. If the source position varies slightly, the signals received at the receiver will be very correlated since the wavelength is very large. Thus, the Fourier transform of their correlation function will have a small support. When the temporal frequency of the emitted signals increases, the received signals are less and less correlated even for small variation of the source position. Therefore the PSD of the SPAF gets a larger support for higher temporal frequencies.

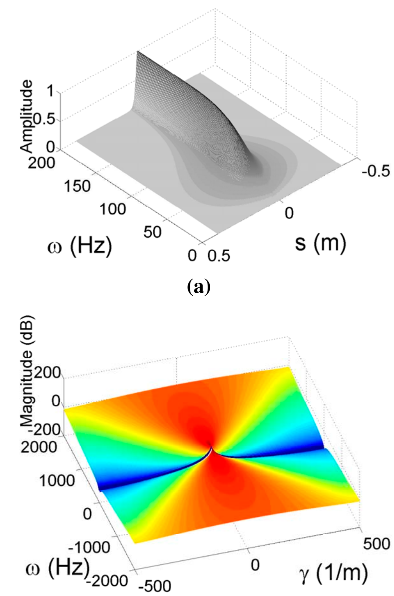


Fig. 3. Study of the SPAF in free field. (a) $R_z(s,\omega)$ as obtained from (20) with eigenvalues at -10 and -100. (b) PSD of this SPAF.

3.2. Influence of the poles of the AR system

By changing the poles (or eigenvalues) of the system, one can control the smoothness of the trajectory. This modifies the opening of the bow-tie shaped spectrum of Fig. 3(b). The smaller the poles are (in absolute value), the more narrow is the spectrum, while for very large values of the poles (in absolute value), the frequency support of the PSD of the SPAF gets very wide. The variation of the spatial bandwidth of the PSD in function of the position of the pole has been studied in the case of an AR1 system. The pole of the system has been changed from $-.5$ to 32. For each of these poles a cut of the PSD of the SPAF was shown at a temporal frequency of 500 Hz in Fig. 4.

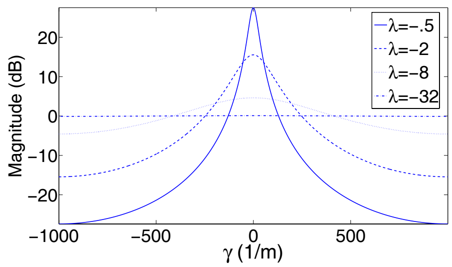


Fig. 4. Comparison at the same temporal frequency for the PSD of a SPAF for different values of the poles of the AR1 system.

3.3. Influence of the order of the AR system

As known from system theory [4], for a fixed position of the poles, increasing the order of an AR model leads to smoother trajectories. Also, for smoother trajectories it is observed that the PSD of the SPAF decays faster. In Fig. 5, a cut of the PSD of a SPAF was considered at a temporal frequency of 500 Hz. The AR models of the three first orders are compared. The poles are all around -5. It can be seen that the AR3 model decays the fastest.

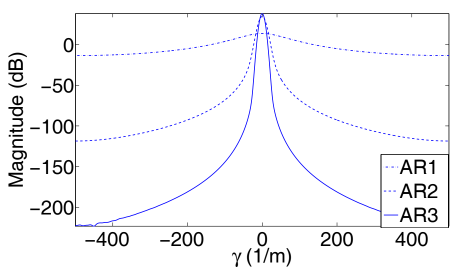
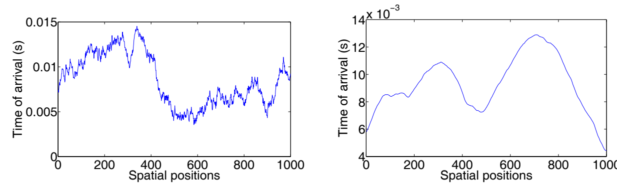


Fig. 5. Comparison at the same temporal frequency for the PSD of a SPAF for AR sytems of different orders.

4. SIMULATIONS AND EXPERIMENTAL MEASUREMENTS

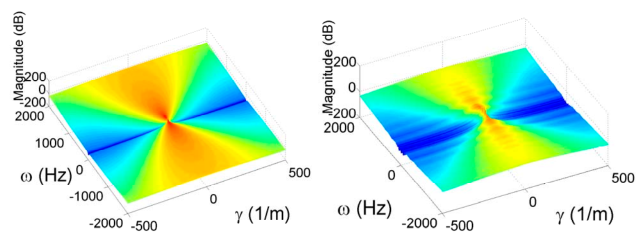
In this section, we present some results obtained by simulations of discrete AR processes. First, simulations were done to show that the higher the order of the AR process, the smoother the trajectory is, as was already discussed in Section. 3.3. A smoother trajectory relates into a smoother time of arrival from the source to the receiver. In the case of an AR1 process, it can be seen in Fig. 6(a) that with low pass systems (pole of the system at -5) the time of arrivals are not very smooth. With an AR2 process with two poles at -5, the time of arrival get smoother as shown in Fig. 6(b). Further, simulations were done to add the effect of reflections introduced by reverberation. We still consider the setup of Fig. 2 but the source and receiver are inside a room. Next to the direct path, reflections on the walls need to be considered. These reflections can be seen as virtual sources. It can be shown that the virtual sources located along the axis of the source and the receiver (axis x in Fig. 2) have a similar PSD as the one of the original source. The other virtual sources being further from the receiver and not on the line source-receiver can be shown to have a narrower frequency support. Therefore the frequency support of the PSD corresponding to the reverberant field is included in the support of the free field PSD. In the simulations, we put the source far enough away



(a) **(b)**

Fig. 6. Smoothness of the AR processes. (a) AR1 process simulated with a discrete pole at -5. (b) AR2 process simulated with two poles around -5.

from the source as in our model and simulated in Fig. 7(a) the PSD of the SPAF in free field and in Fig. 7(b) for a reverberant room.



(a) **(b)**

Fig. 7. PSD of SPAF. (a) For a free field case. (b) For a reverberant room.

5. CONCLUSION

In this paper, we have presented a stochastic model for a spatially varying channel in the case of a source moving along a random trajectory with respect to a fixed receiver. We have modeled the trajectory of the source as an autoregressive model where the poles of the system control the smoothness of the path. Theoretical results are presented for the AR2 case and generalized to any ARn systems. The power spectral density corresponding to the channel impulse response as a function of temporal and spatial frequency has been studied. Simulations results have been shown and compared to the presented theory. This theoretical study gives a basis for a stochastic model of time-varying channels as found in communication environments where a physical process (e.g. user movement) drives a time-varying impulse response of a channel.

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