HIGH RESOLUTION VECTOR-SENSOR ARRAY PROCESSING BASED ON BIQUATERNIONS

Sebastian Miron, Nicolas Le Bihan and Jérôme I. Mars

Laboratoire des Images et des Signaux,
CNRS UMR 5083,
961 Rue de la Houille Blanche,
Domaine Universitaire, BP 46,
38402 Saint Martin d'Hères Cedex,
FRANCE

ABSTRACT

This paper presents a version of MUSIC algorithm for linear vector-sensor arrays based on a complexified quaternionic (biquaternionic) modelization of the output *three-components* vector-signals. A way of computing the eigenvalue decomposition of a biquaternion valued matrix is introduced and the subspace decomposition of the biquaternionic spectral matrix of the observations is used to define the biquaternionic MUSIC estimator (BQ-MUSIC). Performances of the BQ-MUSIC are compared with classical *long-vector* technique.

1. INTRODUCTION

The use of vector-sensors in linear arrays is common in several application areas such as seismic, seismology, electromagnetics or communications. Vector-sensors collect polarized waves and output signals with vector valued samples. In the case of vector-sensors recording the propagating waves in three orthogonal directions, the output vector-signal has three components. These components are linked in phase and amplitude and these relations are driven by the so called *polarization* parameters.

Polarized signals recorded on vector-sensor arrays have been studied with array processing techniques [1]. However, in the literature, it is common use to concatenate the three components of a vector-signal into a three times longer signal with scalar valued samples. This technique is called *long-vector* and has the advantage to extend classical matrix based processing to polarized signals. However, this technique does break the polarization dimension of the signal and does not allow to take advantage of this additional information, without a sophisticated parametrization of the concatenated signals. Also, the long-vector structure may be affected during the process (large datasets) and may lead to impossible recovery of the polarization parameters [2].

In this article, we propose a biquaternion model for three-component vector-signals that allows direct extension of high resolution array processing techniques such as MUSIC [3]. Providing that we define the linear algebra tools needed for decomposition of the observed biquaternionic data into orthogonal subspaces, a biquaternionic version of MUSIC (BQ-MUSIC) can be defined. We introduce this new algorithm and present simulations that emphasize its advantages versus the long-vector approach.

2. BIQUATERNIONIC MODEL FOR VECTOR-SENSOR SIGNALS

2.1. Biquaternions

Biquaternions were discovered, like quaternions, by Sir W.R. Hamilton [4]. Biquaternions are also called *complex quaternions* and, unlike quaternions, do not form a normed division algebra [5]. The set of biquaternions is noted $\mathbb{H}_{\mathbb{C}}$. A biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ is given by:

$$q = q_0 + q_1\mathtt{i} + q_2\mathtt{j} + q_3\mathtt{k} \tag{1}$$

where $q_\alpha \in \mathbb{C}^I$, *i.e.* $q_\alpha = \Re(q_\alpha) + I\Im(q_\alpha)$ with $I = \sqrt{-1}$, and where $\mathtt{i}$, $\mathtt{j}$ and $\mathtt{k}$ are the classical quaternion operators [5]. Like quaternions, biquaternions multiplication is non-commutative ($pq \neq qp$ in general for $p, q \in \mathbb{H}_{\mathbb{C}}$). It is possible to consider a biquaternion as the sum of a scalar and a vector part, both complex valued, such as: $q = \mathcal{S}(q) + \mathcal{V}(q)$ where $\mathcal{S}(q) = q_0$ and $\mathcal{V}(q) = q_1\mathtt{i} + q_2\mathtt{j} + q_3\mathtt{k}$. Also, any biquaternion can be seen in terms of a real and an imaginary part, both being quaternion valued, such as: $q = \Re(q) + I\Im(q)$ where $\Re(q) = \Re(q) + \Re(q_1)\mathtt{i} + \Re(q_2)\mathtt{j} + \Re(q_3)\mathtt{k}$ and $\Im(q) = \Im(q) + \Im(q_1)\mathtt{i} + \Im(q_2)\mathtt{j} + \Im(q_3)\mathtt{k}$. A biquaternion with null scalar part is called *pure*. Note that any complex number $z \in \mathbb{C}^I$ commutes with any biquaternion $q \in \mathbb{H}_{\mathbb{C}}$ ($zq = qz$).

There exist different conjugates for a biquaternion q: the *complex conjugate* $q^{\triangleleft} = q_0^* + q_1^*\mathtt{i} + q_2^*\mathtt{j} + q_3^*\mathtt{k}$, the *quaternion conjugate* $\overline{q} = q_0 - q_1\mathtt{i} - q_2\mathtt{j} - q_3\mathtt{k}$ and the *Hermitian conjugate* $q^* = (\overline{q})^{\triangleleft} = q_0^* - q_1^*\mathtt{i} - q_2^*\mathtt{j} - q_3^*\mathtt{k}$. Note that Hermitian conjugation for biquaternions is an anti-involution: $(qp)^* =$

p^*q^*, for $q, p \in \mathbb{H}_\mathbb{C}$. The norm of a biquaternion can be defined as : $|q| = \sqrt{<q,q>} = \sqrt{|q_0|^2 + |q_1|^2 + |q_3|^2 + |q_3|^2}$ where the *scalar product* of two biquaternions is: $<q,p>= q_0^* p_0 + q_1^* p_1 + q_2^* p_2 + q_3^* p_3$.

Also, we introduce here the concept of correlation between two biquaternions[1] and random biquaternion valued vector. Such a vector, with N entries, has values in $\mathbb{H}_\mathbb{C}^N$. A scalar product can be defined for two N-dimensional quaternion valued vectors $\mathbf{v}$ and $\mathbf{w} \in \mathbb{H}_\mathbb{C}^N$ such as: $<\mathbf{u}, \mathbf{v}>_{\mathbb{H}_\mathbb{C}} = \mathbb{E}[\mathbf{u}^\dagger \mathbf{v}]$. Orthogonality between two biquaternion valued vectors can be stated when their scalar product is null, *i.e.* $<\mathbf{u}, \mathbf{v}>_{\mathbb{H}_\mathbb{C}} = 0$.

We now propose to use biquaternion valued signals to model polarized signals recorded on vector-sensor array.

2.2. Vector-sensor array signal model

2.2.1. Polarized three-component signal as biquaternion valued signal

Consider a vector-sensor with output signal $s(t)$. Each one of the three components records a signal in an orthogonal direction: $s_1(t), s_2(t)$ and $s_3(t)$. These signals are correlated if the wave that generated them is polarized. In this case, the three signals are linked by phase and amplitude coefficients called *polarization parameters* (supposed constant in time and frequency). Considering that the three signals in time are transformed in the Fourier domain as: $s_\eta(\nu) = \mathtt{FT}\left[s_\eta(t)\right]$, with $s_\eta(\nu) \in \mathbb{C}^I$ and $\eta = 1, 2, 3$, then the *three-component* signal recorded on a vector-sensor can be expressed in the frequency domain such as:

$$s(\nu) = s_1(\nu)\mathtt{i} + s_2(\nu)\mathtt{j} + s_3(\nu)\mathtt{k} \tag{2}$$

This signal is *pure* biquaternion valued. Taking the first component $s_1(\nu)$ as reference and with the previous assumptions on polarization parameters, $s(\nu)$ can be rewritten as:

$$s(\nu) = \left(\mathtt{i} + \rho_1 e^{I\varphi_1}\mathtt{j} + \rho_2 e^{I\varphi_2}\mathtt{k}\right) s_1(\nu) \tag{3}$$

where ρ_1, ρ_2 and φ_1, φ_2 are the amplitude ratios and the phase-shifts for the second and the third component respectively, with respect to the first one. In the following, the working frequency will be omitted, assuming that we consider narrow-band signals or that we work independently at each frequency.

2.2.2. Propagation and polarization model

Now, considering a set of N equally-spaced vector-sensors, recording the contribution of L polarized plane waves, the recorded signal $\mathbf{x} \in \mathbb{H}_\mathbb{C}^N$ is given as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \sum_{l=1}^{L} p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l}) \mathbf{a}(\theta_l) s_l + \mathbf{b} \tag{4}$$

[1]Consider two pure biquaternions p, q, then $\mathbb{E}[pq^*] = \mathbb{E}[p_1 q_1^* + p_2 q_2^* + p_3 q_3^*] + \mathbb{E}[p_3 q_2^* - p_2 q_3^*]\mathtt{i} + \mathbb{E}[p_1 q_3^* - p_3 q_1^*]\mathtt{j} + \mathbb{E}[p_2 q_1^* - p_1 q_2^*]\mathtt{k}$.

where $p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l})$ is the biquaternion valued polarization coefficient of the l^{th} wave containing its polarization parameters, $\mathbf{a}(\theta_l)$ is the propagation vector of the l^{th} wave on the array and is given (assuming plane waves contributions only) by:

$$\mathbf{a}(\theta_l) = \begin{bmatrix} 1 & e^{-I\theta_l} & \dots & e^{-I(N-1)\theta_l} \end{bmatrix}^T \tag{5}$$

The vector $\mathbf{b} \in \mathbb{H}_\mathbb{C}^N$ contains unpolarized noise contributions on the vector-sensor array. Also, the s_l coefficients correspond to the magnitude contribution of the l^{th} wave (at a fixed frequency). In the following, we use the notation:

$$\mathbf{d}_l(\theta_l, \rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l}) = p_l(\rho_{1l}, \varphi_{1l}, \rho_{2l}, \varphi_{2l}) \mathbf{a}(\theta_l) \tag{6}$$

so that $\mathbf{d}_l$ is called the *polarized steering vector* of the l^{th} wave and so that the observations can be written as:

$$\mathbf{x} = \sum_{l=1}^{L} \mathbf{d}_l s_l + \mathbf{b} \tag{7}$$

The biquaternion observation vector $\mathbf{x} \in {\mathbb{H}_\mathbb{C}}^N$ is built from the observations on the three components as: $\mathbf{x} = \mathbf{x}_1\mathtt{i} + \mathbf{x}_2\mathtt{j} + \mathbf{x}_3\mathtt{k}$. As a comparison, in the long-vector approach, the three observations are concatenated in a complex vector $\mathbf{x} \in \mathbb{C}^{3N}$:

$$\mathbf{x}_{LV} = \left[\mathbf{x}_1^T | \mathbf{x}_2^T | \mathbf{x}_3^T\right]^T. \tag{8}$$

This biquaternion model is now used to define a version of MUSIC algorithm for three-component vector-signals.

3. BQ-MUSIC ESTIMATOR

The MUSIC algorithm is based on the decomposition of the biquaternionic spectral matrix of the observation data vector $\mathbf{x}$ into signal and noise orthogonal subspaces.

3.1. Biquaternionic spectral matrix

3.1.1. Definition

If the output of the vector-sensor array is $\mathbf{x} \in \mathbb{H}_\mathbb{C}^N$ given in (4), then the spectral matrix is defined as:

$$\mathbf{\Lambda} = \mathbb{E}[\mathbf{x}\mathbf{x}^\dagger] \in {\mathbb{H}_\mathbb{C}}^{N \times N} \tag{9}$$

Assuming decorrelation between the different sources and between sources and noise, the biquaternionic spectral matrix takes the following form:

$$\mathbf{\Lambda} = \sum_{l=1}^{L} \sigma_l^2 \mathbf{d}_l \mathbf{d}_l^\dagger + \mathbf{\Lambda_b} \tag{10}$$

where σ_l are the powers of the L sources on the antenna and $\mathbf{d}_l \in {\mathbb{H}_\mathbb{C}}^N$ are the *biquaternionic source vectors* describing source contributions on the antenna. The matrix $\mathbf{\Lambda}_b$ is given by: $\mathbf{\Lambda_b} = \mathbb{E}[\mathbf{b}\mathbf{b}^\dagger] = diag(\sigma_1^2, \sigma_2^2, \dots, \sigma_N^2)$, where

$\sigma_n^2 = \mathbb{E}[b_n b_n^*]$ is the power of the noise on the n^{th} sensor. In order to build a MUSIC estimator, it is necessary to decompose the observation data spectral matrix. We propose for the first time, in subsection 3.2, an algorithm to achieve this decomposition.

3.1.2. Computational issues

If the three-component *long-vector* model is used (8), the spectral matrix is complex of size $3N \times 3N$. Compared to this long-vector matrix having $9N^2$ complex entries, the spectral matrix in the biquaternionic approach has N^2 biquaternion-valued coefficients. As a biquaternion is composed of 4 complex numbers, the biquaternion spectral matrix can thus be represented on $4N^2$ complex values. This way, the memory requirements for data covariance representation are reduced by a factor of $4/9$, provided that a biquaternion model is used.

3.2. Biquaternionic matrices diagonalization

Matrices with biquaternionic coefficients (*i.e.* elements of $\mathbb{H}_{\mathbb{C}}^{N\times M}$) were not paid much attention and most of the known results are given in [6]. A biquaternion matrix $\mathbf{A} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ is said *Hermitian* if $\mathbf{A} = \mathbf{A}^\dagger$, where $\dagger$ is the transposition-(Hermitian) conjugation operator. In order to propose a diagonalization algorithm for Hermitian biquaternionic matrices, we introduce the *quaternion adjoint matrix* of a biquaternion valued matrix.

3.2.1. Quaternion adjoint matrix

Any biquaternion valued matrix $\mathbf{C} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ can be written as: $\mathbf{C} = \mathbf{C}_1 + I\mathbf{C}_2$, where $\mathbf{C}_1, \mathbf{C}_2 \in \mathbb{H}^{N\times N}$. Then, the *quaternion adjoint* matrix, noted $\Upsilon_{\mathbf{C}} \in \mathbb{H}^{2N\times 2N}$, associated to $\mathbf{C}$ is defined as:

$$\Upsilon_{\mathbf{C}} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ -\mathbf{C}_2 & \mathbf{C}_1 \end{bmatrix} \tag{11}$$

It can be demonstrated that if $\mathbf{C}$ is Hermitian, then so is $\Upsilon_{\mathbf{C}}$ and if $\mathbf{C}$ is unitary, then $\Upsilon_{\mathbf{C}}$ is also unitary.

3.2.2. Biquaternionic EVD

Just like in the quaternion case, there exists two kinds of eigenvalues for biquaternion matrices: the *left* and *right* eigenvalues [7]. Here, we only consider right eigenvalues. This is motivated by the fact that we can link them to the right eigenvalues of the *quaternion adjoint matrix* and that only the right eigenvalues of quaternion matrices have a completely known theory up to now [7].

Consider a Hermitian biquaternion valued matrix $\mathbf{C} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ and its *quaternion adjoint matrix* $\Upsilon_{\mathbf{C}} \in \mathbb{H}^{2N\times 2N}$. As $\Upsilon_{\mathbf{C}}$ is Hermitian, there exists a quaternion unitary matrix $\mathbf{V} \in \mathbb{H}^{2N\times 2N}$ such that: $\mathbf{V}^\dagger \Upsilon_{\mathbf{C}} \mathbf{V} = diag(\delta_1, \delta_2, \ldots, \delta_{2N})$ where $\delta_p \in \mathbb{R}, \forall p$ (see [7]). This implies that matrix $\mathbf{C}$ can be decomposed as:

$$\mathbf{C} = \mathbf{U}\boldsymbol{\Delta}\mathbf{U}^\dagger \tag{12}$$

where $\mathbf{U} \in \mathbb{H}^{N\times 2N}$, with the property $\mathbf{U}^\dagger\mathbf{U} = \mathbf{I}$, contains the 2N biquaternionic eigenvectors and $\Delta \in \mathbb{R}^{2N\times 2N}$ contains the real eigenvalues of $\mathbf{C}$. Also, this eigenvalue decomposition of a Hermitian biquaternionic matrix can be written as:

$$\mathbf{C} = \sum_{p=1}^{2N} \mathbf{u}_p \mathbf{u}_p \lambda_p \tag{13}$$

where the 2N *eigenvectors* of $\mathbf{C}$ form an orthonormal basis over $\mathbb{H}_{\mathbb{C}}^{N}$. Note that a $N \times N$ biquaternion-valued matrix has $2N$ different eigenvalues and $2N$ independent eigenvectors. This means that a source present in the signal is represented by two eigenvalues in the decomposition of the biquaternion spectral matrix. The orthogonality constraint between the $\mathbf{u}_p$ is governed by the biquaternionic scalar product and it can be shown that it implies more restrictive relationships between the components of the array than the *long-vector* model.

3.3. BQ-MUSIC estimator

As presented in (6), every polarized wave impinging on the vector-sensor array has five parameters and the proposed version of MUSIC intends to estimate the five of them simultaneously. In order to do so, and as usual in MUSIC approach, a parametrized steering vector is projected onto the *noise* subspace built using the last eigenvectors of the spectral matrix of the observations. The biquaternionic steering vector has the following expression:

$$\mathbf{f}(\Omega) = \frac{1}{\mathcal{N}} \begin{bmatrix} z & ze^{-I\theta} & \ldots & ze^{-I(N-1)\theta} \end{bmatrix}^T \tag{14}$$

where $\Omega = \{\theta, \rho_1, \rho_2, \varphi_1, \varphi_2\}$, $z = \mathtt{i} + \rho_1 e^{I\varphi_1}\mathtt{j} + \rho_2 e^{I\varphi_2}\mathtt{k}$ and $\mathcal{N} = \sqrt{N(1 + \rho_1^2 + \rho_2^2)}$. Then, the BQ-MUSIC consists in finding the set of parameters Ω that maximizes the following functional:

$$\mathcal{F}(\Omega) = \frac{1}{\mathbf{f}^\dagger(\Omega)\boldsymbol{\Pi}_B\mathbf{f}(\Omega)} \tag{15}$$

where $\boldsymbol{\Pi}_B = \sum_{p=2L+1}^{2N} \mathbf{u}_p\mathbf{u}_p^\dagger$ is built with the last $2(N-L)$ eigenvectors of $\boldsymbol{\Lambda}$.

4. SIMULATIONS

The BQ-MUSIC algorithm proposed in this paper is compared to the corresponding algorithm LV-MUSIC based on the classical *long-vector* model (8). First, we consider a scenario with one polarized source recorded on an array of ten three-component sensors. The source polarization parameters are $\theta = 0.095\ rad$, $\rho_1 = 1$, $\rho_2 = 2$, $\varphi_1 = 0\ rad$ and $\varphi_2 = 0.35\ rad$. One hundred samples have been used to estimate the interspectral matrices. Fig. 1 plots the functions for BQ and LV algorithms for fixed values of polarization parameters $\rho_1, \rho_2, \varphi_1, \varphi_2$, corresponding to the polarization of the impinging wave. The 3dB width of the detection peak for the BQ algorithm is smaller compared to the LV one, meaning a that the BQ approach presents a better resolution power. Fig. 2 illustrates a two sources case. We considered that the number of sources has been badly estimated and only one source was considered for the computations. BQ outperforms LV when robustness to this kind of errors is considered. These results can be explained by the fact that when diagonalizing the interspectral matrix, a stronger orthogonality constraint is imposed between noise and signal subspaces in the BQ approach. In Fig. 3 the root-mean square error for DOA estimation versus SNR, for BQ and LV (one point = one hundred runs). A small loss of accuracy is observed for the BQ algorithm, especially for very low SNR, but generally the proposed algorithm performs fairly well compared to the LV one. The explanation to this minor drawback is the compression of information in biquaternion interspectral matrix, that reduces memory requirements for data covariance representation, as shown in subsection 3.1.2. However, the proposed model takes into account only the orthogonality and not the coupling between the vector-sensor components.

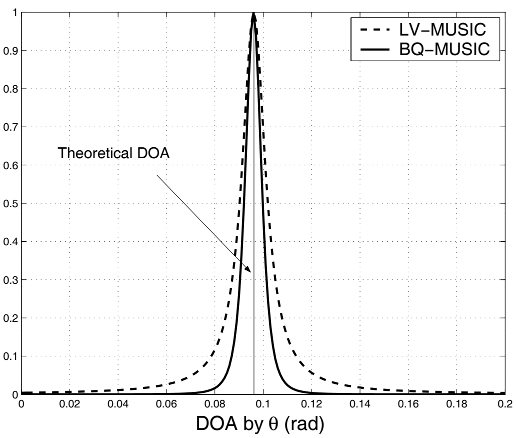

Fig. 1. DOA estimation for a one source case

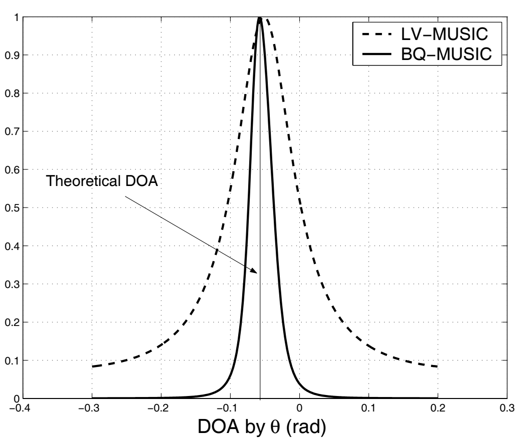

Fig. 2. DOA estimation for sub-estimated number of sources

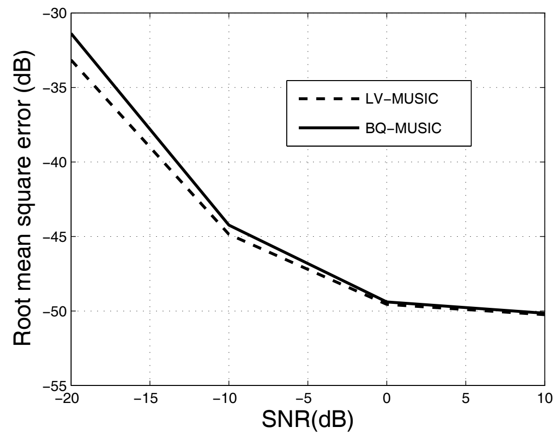

Fig. 3. Root mean square error for DOA estimation

5. CONCLUSIONS

This article presents a biquaternion model for polarized sources recorded on three-component vector-sensor arrays that allows a simple derivation of a MUSIC-like algorithm (BQ-MUSIC). The use of biquaternions reduces by approximately half the memory size required for data covariance representation while it increases resolution for DOA estimation with close, together with almost the same RMSE on DOA as the long-vector-approach.

6. REFERENCES

[1] A. Nehorai and E. Paldi, "Vector-sensor array processing for electromagnetic source localization," *IEEE transactions on signal processing*, vol. 42, no. 2, pp. 376–398, 1994.

[2] D. Rahamim, J. Tabrikian, and Shavit R., "Source localization using vector sensor array in multipath environment," *IEEE Trans. on Signal Processing*, vol. 52, no. 11, pp. 3096–3103, 2004.

[3] R.O. Schmidt, "Multiple emmitter location and signal parameter estimation," *IEEE Trans. on antennas and propagation*, vol. AP-34, no. 3, pp. 276–280.

[4] W.R. Hamilton, "On the geometrical interpretation of some results obtained by calculation with biquaternions," *Proceeding of the Royal Irish Academy*, vol. V, pp. 388–390, 1853.

[5] J.P. Ward, *Quaternions and Cayley Numbers, Algebra and applications*, Kluwer Academic, 1997.

[6] Y. Tian, "Matrix theory over the complex quaternion algebra," *ArXiv Mathematics e-prints*, 2000.

[7] F. Zhang, "Quaternions and matrices of quaternions," *Linear algebra and its applications*, vol. 251, pp. 21–57, 1997.