A ROBUST CAPON BEAMFORMER WITH NEW UNCERTAINTY CONSTRAINT ON STEERING VECTOR

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ABSTRACT

A robust Capon beamformer (RCB) with a new constraint on the uncertainty of nominal array steering vector (ASV) is proposed in this paper. The new constraint is constructed by replacing the nominal ASV with a projected one onto the signal-plus-interference subspace. The proposed RCB achieves higher output signal-to-noise-plus-interference ratio (SINR) compared with the conventional RCBs. Theoretical analysis and simulation results show the effectiveness of the proposed method.

1. INTRODUCTION

Adaptive array has been studied for some decades as an attractive solution to signal detection and estimation in harsh environments. It is widely used in wireless communications, microphone array processing, radar, sonar and medical imaging. A famous representative, named Capon beamformer [1], minimizes array output power subject to a linear constraint, known as look direction constraint, which ensures desired array response from a specific direction. It has high performance in interference suppression if the ASV corresponding to signal-of-interest (SOI) is known accurately.

Some of the underlying assumptions on environment, source or sensor array can be violated when adaptive arrays are used in practical applications. This may cause mismatch between the nominal and the actual ASVs. For adaptive beamformer, ASV mismatch results in target signal cancellation. Some robust beamformers have been proposed to avoid performance degradation due to array imperfections (See [2, 3] and references therein). However, most of these solutions only deal with steering direction error. When ASV mismatch is caused by array perturbation, array manifold mismodelling, or wavefront distortion , these methods cannot achieve sufficient improvement on robustness [4].

In this paper, a RCB is derived by maximizing the output power [5–7] of the standard Capon beamformer (SCB) with respect to all feasible ASVs in uncertainty set [8–11]. The resulting RCB has similar mathematical form as the beamformer in [10, 11]. Since the robustness of the beamformers in [8–11] is obtained at the cost of reduced capability of noise/interference suppression, the performance degradation is serious with large uncertainty set. Hence, compact uncertainty set is appreciated to guarantee the performance on noise/interference suppression. In this paper, we propose a new compact uncertainty set, which is obtained by replacing the nominal ASV in conventional uncertainty set [10, 11] with the projection of the the nominal ASV onto the signal-plus-interference subspace. Theoretical analysis and simulation results show that the SINR improvement of the proposed RCB is higher than that of the RCBs in [10, 11].

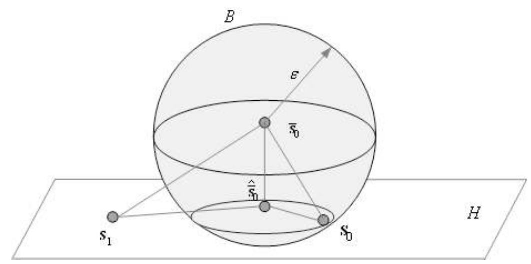

Fig. 1: Geometry model of the uncertainty set

2. PROPOSED METHOD

2.1. Derivation of New Compact Uncertainty Constraint

Assume there are K directional signals impinging on the array. The eigenvalue decomposition of the covariance matrix $\mathbf{R}$ of the array snapshot is

$$\mathbf{R} = \mathbf{U}\mathbf{\Gamma}\mathbf{U}^H, \tag{1}$$

where $\mathbf{U}$ is the eigenvector matrix and $\mathbf{\Gamma} = diag\{\lambda_1, \cdots, \lambda_M\}$ is the eigenvalue matrix. The orthogonal bases $\mathbf{U}_s$ of the ASVs $\{\mathbf{s}_k\}$ are obtained by extracting the eigenvectors corresponding to the largest K eigenvalues. $\mathbf{U}_s$ spans a linear space H,

$$H = \{\mathbf{s}|\mathbf{s} = \mathbf{U}_s\mathbf{c},\ \mathbf{c} \in \mathbb{C}^K\}, \tag{2}$$

where $\mathbb{C}^K$ is K-dimensional complex vector space. Assume that there are two signals. Without loss of generality, H is illustrated in 3-dimension as shown in Fig. 1. Herein, the nominal ASV $\bar{\mathbf{s}}_0$ does not coincide with the actual one $\mathbf{s}_0$. Although the actual ASV $\mathbf{s}_0$ is unknown, it locates in the space H. With this property, a new nominal ASV $\hat{\bar{\mathbf{s}}}_0$ in H can be estimated to form a compact uncertainty set with lower uncertainty level. This new ASV $\hat{\bar{\mathbf{s}}}_0$ is designed as a vector in H and nearest to $\bar{\mathbf{s}}_0$. It can be expressed as

$$\hat{\bar{\mathbf{s}}}_0 = \mathbf{U}_s\hat{\mathbf{c}}, \quad \hat{\mathbf{c}} = arg\ \min_{\mathbf{c}} ||\mathbf{U}_s\mathbf{c} - \bar{\mathbf{s}}_0||_2^2, \tag{3}$$

where $||\cdot||_2$ represents the Euclidean norm. The optimal solution of $\hat{\bar{\mathbf{s}}}_0$ is

$$\hat{\bar{\mathbf{s}}}_0 = \mathbf{U}_s\mathbf{U}_s^H\bar{\mathbf{s}}_0. \tag{4}$$

Since $\hat{\bar{\mathbf{s}}}_0$ is the projection of $\bar{\mathbf{s}}_0$ onto the signal-plus-interferences subspace H, it is straightforward that the distance between the estimated ASV $\hat{\bar{\mathbf{s}}}_0$ and the actual one $\mathbf{s}_0$ is shorter than that between $\bar{\mathbf{s}}_0$ and $\mathbf{s}_0$. The new uncertainty constraint can be formulated as

$$||\mathbf{s} - \hat{\bar{\mathbf{s}}}_0||_2^2 \leq \epsilon', \tag{5}$$

where ϵ' is the new uncertainty level. If $\hat{\bar{\mathbf{s}}}_0$ in (5) is replaced by $\bar{\mathbf{s}}_0$, the uncertainty constraint in (5) is similar to the one in [10, 11].

2.2. Proposed Robust Capon Beamformer

If the steering vector $\mathbf{s}_0$ of the SOI is known, the Capon beamformer is formulated as the following linearly constrained quadratic optimization problem.

$$\min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{s.t.} \ \ \mathbf{s}_0^H \mathbf{w} = 1, \tag{6}$$

where $\mathbf{w}$ is the weight vector of the beamformer. The optimal weight $\mathbf{w}_0$ and output power $\hat{\sigma}_s^2$ of (6) is

$$\mathbf{w}_0 = \frac{\mathbf{R}^{-1}\mathbf{s}_0}{\mathbf{s}_0^H \mathbf{R}^{-1}\mathbf{s}_0}, \ \hat{\sigma}_s^2 = \frac{1}{\mathbf{s}_0^H \mathbf{R}^{-1}\mathbf{s}_0}. \tag{7}$$

In real world, the ASV $\mathbf{s}_0$ is always unknown or known but with some error. If the nominal ASV deviates from the true one, target signal cancellation occurs in the Capon beamformer, and the output power $\hat{\sigma}_s^2$ in (7) decreases. A method to overcome the problem of signal cancellation is to search for an optimal ASV $\mathbf{s}$ which results in maximal output power $\hat{\sigma}_s^2$. We assume that the true ASV $\mathbf{s}_0$ belongs to the uncertainty constraint set (5), the robust beamformer can be formulated as

$$\max_{\mathbf{s}} \min_{\mathbf{w}} \ \mathbf{w}^H \mathbf{R} \mathbf{w} \quad \text{s.t.} \ \ \mathbf{s}^H \mathbf{w} = 1, \ \ ||\mathbf{s} - \hat{\bar{\mathbf{s}}}_0||_2^2 \le \epsilon'. \tag{8}$$

This optimization problem can be solved in two steps. First, we fix $\mathbf{s}$ and search for the minimal output power. Then we search for the maximal value of the minimal output power to all the possible $\mathbf{s}$. For any given $\mathbf{s}$, The optimization problem in (8) is simplified to

$$\min_{\mathbf{s}} \ \mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} \quad \text{s.t.} \ \ ||\mathbf{s} - \hat{\bar{\mathbf{s}}}_0||_2^2 \le \epsilon'. \tag{9}$$

The optimization problem (9) can be solved using the Lagrange multiplier methodology [12] Its optimal solution of (9) is obtained on the boundary of the constraint set. The optimization problem in (9) can be reformulated as

$$\min_{\mathbf{s}} \ \mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} \quad s.t. \ ||\mathbf{s} - \hat{\bar{\mathbf{s}}}_0||_2^2 = \epsilon'. \tag{10}$$

To exclude the trivial solution $\mathbf{s} = \mathbf{0}$ to (9), we assume that

$$||\hat{\bar{\mathbf{s}}}_0||_2^2 \ge \epsilon'. \tag{11}$$

Define a function

$$f = \mathbf{s}^H \mathbf{R}^{-1} \mathbf{s} + g(||\mathbf{s} - \hat{\bar{\mathbf{s}}}_0||_2^2 - \epsilon'), \tag{12}$$

where $g \ge 0$ is the Lagrange multiplier. The optimal vector $\hat{\mathbf{s}}$ is obtained by setting the differentiation of (12) with respect to $\mathbf{s}^*$ as zero.

$$\frac{df}{d\mathbf{s}^*} = \hat{\mathbf{R}}^{-1}\hat{\mathbf{s}} + g(\hat{\mathbf{s}} - \hat{\bar{\mathbf{s}}}_0) = \mathbf{0}, \tag{13}$$

where $(\cdot)^*$ is the conjugate operator. The above equation yields

$$\hat{\mathbf{s}} = (g^{-1}\hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1}\hat{\bar{\mathbf{s}}}_0 = \hat{\bar{\mathbf{s}}}_0 - (\mathbf{I} + g\hat{\mathbf{R}})^{-1}\hat{\bar{\mathbf{s}}}_0, \tag{14}$$

where $\mathbf{I}$ is the identity matrix. The Lagrange multiplier g is the root of the constraint equation

$$||\hat{\mathbf{s}} - \hat{\bar{\mathbf{s}}}_0||_2^2 = ||(\mathbf{I} + g\hat{\mathbf{R}})^{-1}\hat{\bar{\mathbf{s}}}_0||_2^2 = \epsilon'. \tag{15}$$

It can be proved that there is a unique solution $\hat{g} \ge 0$ of (15) [10, 11]. So that

$$\hat{\mathbf{s}} = (\hat{g}^{-1}\hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1}\hat{\bar{\mathbf{s}}}_0 = \hat{\bar{\mathbf{s}}}_0 - (\mathbf{I} + \hat{g}\hat{\mathbf{R}})^{-1}\hat{\bar{\mathbf{s}}}_0. \tag{16}$$

The corresponding optimal weight of the robust beamformer and its output SINR are given by

$$\mathbf{w}_0 = \frac{\hat{\mathbf{R}}^{-1}\hat{\mathbf{s}}}{\hat{\mathbf{s}}^H \hat{\mathbf{R}}^{-1}\hat{\mathbf{s}}}, \ \ \rho = \frac{\mathbf{w}_0^H \hat{\mathbf{R}}_s \mathbf{w}_0}{\mathbf{w}_0^H \hat{\mathbf{R}}_n \mathbf{w}_0}. \tag{17}$$

3. ANALYSIS OF OUTPUT SINR

In this section, an analysis of the output SINR of the proposed RCB (PRCB) and the conventional RCB is carried out. Since a complete analysis of SINR performance under general array imperfections represents a formidable analytical task, in this paper, a simplified problem is discussed. We assume that only steering vector error exists in the array processor and the theoretical covariance matrix is used. In such case, the performance degradation of the Capon beamformer is caused by the error in the nominal ASV.

When there is only ASV error, a general conclusion on the output SINR of the Capon beamformer is given in Lemma 1.

Lemma 1. *Assume that the covariance matrices of the SOI and the interference/noise are $\mathbf{R}_s$ and $\mathbf{R}_n$, respectively. The covariance matrix of array snapshot is $\mathbf{R} = \mathbf{R}_s + \mathbf{R}_n$. When the nominal ASV is given by $\mathbf{s}$, and the true ASV is $\mathbf{s}_0$. The output SINR ρ of the Capon beamformer is*

$$\rho = \frac{\rho_o \cos^2(\theta)}{1 + \sin^2(\theta)\rho_o(\rho_o + 2)}, \tag{18}$$

where θ is the extended angle between vector $\mathbf{s}$ and $\mathbf{s}_0$, and ρ_o is the output SINR of the Capon beamformer when $\mathbf{s}_0$ is known.

$$\cos^2(\theta) = \frac{|\mathbf{s}_0^H \mathbf{R}_n^{-1}\mathbf{s}|^2}{||\mathbf{s}_0||_{\mathbf{R}}^2 ||\mathbf{s}||_{\mathbf{R}}^2}, \quad \rho_o = \sigma_s^2 \mathbf{s}_0^H \mathbf{R}_n^{-1}\mathbf{s}_0 = \sigma_s^2 ||\mathbf{s}_0||_{\mathbf{R}}^2, \tag{19}$$

where $||\mathbf{x}||_{\mathbf{R}}^2 \triangleq \mathbf{x}^H \mathbf{R}_n^{-1}\mathbf{x}$ is the extended vector norm ($\mathbf{R}_n$ is a positive matrix), and σ_s^2 is the power of the SOI. If $\mathbf{R}_n = \sigma_n^2 \mathbf{I}$, the extended vector norm $||\cdot||_{\mathbf{R}}$ can be replaced by the Euclidian norm $||\cdot||_2$, and

$$\cos^2(\theta) = \frac{|\mathbf{s}_0^H \mathbf{s}|^2}{||\mathbf{s}_0||_2^2 ||\mathbf{s}||_2^2}, \quad \rho_{opt} = \frac{\sigma_s^2}{\sigma_n^2}||\mathbf{s}_0||_2^2. \tag{20}$$

Proof. Refer to [13]. □

Lemma 1 indicates that the output SINR of the Capon beamformer is determined by the angle between the nominal ASV and the true one. Moreover, it is easy to show that the output SINR ρ is a monotonically increasing function of $\cos^2(\theta)$. The PRCB and the RCB have similar mathematical form as the Capon beamformer except that the nominal vector $\mathbf{s}$ is replaced by the estimated one, $\hat{\bar{\mathbf{s}}}_0$ or $\bar{\mathbf{s}}_0$. Therefore, the output SINR of the PRCB (RCB) can be analyzed via the angle between $\hat{\bar{\mathbf{s}}}_0$ ($\bar{\mathbf{s}}_0$) and $\mathbf{s}_0$.

Lemma 2. *The ASVs used in calculation of array optimal weight for the conventional RCB [11] and the PRCB are $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. According to (9) and (14), we have*

$$\mathbf{s}_1 = (g_1^{-1}\hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1}\bar{\mathbf{s}}_0, \quad \mathbf{s}_2 = (g_2^{-1}\hat{\mathbf{R}}^{-1} + \mathbf{I})^{-1}\hat{\bar{\mathbf{s}}}_0, \tag{21}$$

where the scales g_1 and g_2 are the optimal diagonal loading factors. Denoting

$$\cos^2(\theta_1) = \frac{||\mathbf{s}_0^H \mathbf{s}_1||_{\mathbf{R}}^2}{||\mathbf{s}_0||_{\mathbf{R}}^2 ||\mathbf{s}_1||_{\mathbf{R}}^2}, \quad \cos^2(\theta_2) = \frac{||\mathbf{s}_0^H \mathbf{s}_2||_{\mathbf{R}}^2}{||\mathbf{s}_0||_{\mathbf{R}}^2 ||\mathbf{s}_2||_{\mathbf{R}}^2}, \tag{22}$$

we have

$$\cos^2(\theta_1) \le \cos^2(\theta_2). \tag{23}$$

Proof. Refer to Appendix A. □

According to Lemma 1 and 2, it can be concluded that the output SINR ρ_2 of the PRCB is higher than that of the conventional RCB ρ_1, i.e.,

$$\rho_2 \ge \rho_1. \tag{24}$$

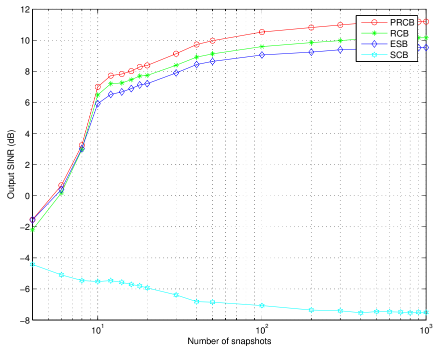

Fig. 2: Performance comparison of the beamformers with 3 impinging sources

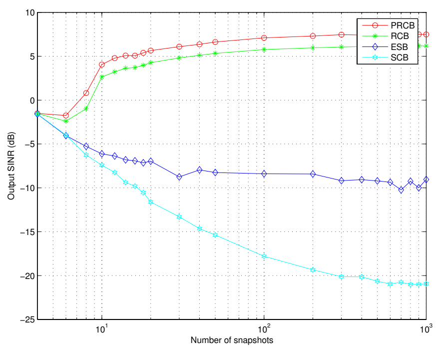

Fig. 3: Performance comparison of the beamformers with 7 impinging sources

4. NUMERICAL STUDY

In this section, some numerical experiments were carried out to evaluate the performance of the proposed RCB. An uniform linear array containing ten sensors with half-wavelength spacing is used to enhance the SOI in the presence of strong interferences as well as the uncertainty in the ASV. In all experiments, the nondirectional noise is a spatially white Gaussian noise with unit covariance. The array covariance matrix is estimated with different number of snapshots N. The output SINR is the average of 200 Monte-Carlo experiments. The performance of the SCB [1], the RCB [10], and the eigenspace-based beamformer (ESB) [14] are also included for the purpose of performance comparison.

The assumed direction-of-arrival (DOA) of the SOI is $\theta_0 = 0^\circ$. The DOA and the power of the SOI is $(6^\circ, 10dB)$. The DOAs and the powers of two interferences are $(60^\circ, 20dB)$ and $(80^\circ, 20dB)$, respectively. The results in Fig. 2 show that the PRCB has higher output SINR than others.

It is known that the performance of the projection based method degrades when the dimension of signal-plus-interference subspace is high. In next experiment, we include additional four interferences, whose DOAs and powers are $(-30^\circ, 20dB)$, $(-50^\circ, 10dB)$, $(-70^\circ, 20dB)$ and $(-85^\circ, 20dB)$, respectively. The results shown in Fig. 3 clearly indicate that the ESB has poor performance because of large error in the projected ASV due to the high dimension of signal-plus-interference subspace. The PRCB outperforms the other beamformers in output SINR. It is also known that the performance of the ESB strongly depends on the accurate knowledge of the dimension of signal-plus-interference subspace. The problem can be solved by using overestimate the dimension of of signal-plus-interference for PRCB, as shown in Fig. 4.

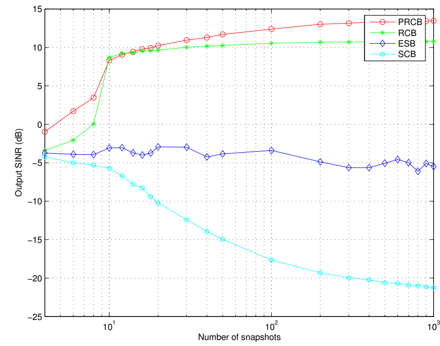

Fig. 4: Comparison of the output SINR of the beamformers versus number of snapshots with 3 impinging sources (The dimension of signal-plus-interference subspace is overestimated as 4.)

5. CONCLUSION

A robust Capon beamformer with a new constraint on the uncertainty of array steering vector is proposed in this paper. It is robust to arbitrary error in array steering vector and demonstrates superior performance on SINR improvement. Simulations results and theoretical analysis show that the proposed RCB outperforms the conventional RCB in the output SINR.

A. PROOF OF LEMMA 2

Proof. Using the eigen-decomposition of $\mathbf{R}$, we have

$$\begin{aligned}\mathbf{s}_1 &= (g_1^{-1}\mathbf{R}^{-1} + \mathbf{I})^{-1}\bar{\mathbf{s}}_0 \\ &= \mathbf{U}\begin{bmatrix} \frac{g_1\lambda_1}{1+g_1\lambda_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \frac{g_1\lambda_M}{1+g_1\lambda_M} \end{bmatrix}\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &\triangleq \mathbf{U}\mathbf{D}_1\mathbf{U}^H\bar{\mathbf{s}}_0,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\mathbf{s}_2 &= (g_2^{-1}\mathbf{R}^{-1} + \mathbf{I})^{-1}\hat{\bar{\mathbf{s}}}_0 \\ &= \mathbf{U}\begin{bmatrix} \frac{g_2\lambda_1}{1+g_2\lambda_1} & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \frac{g_2\lambda_M}{1+g_2\lambda_M} \end{bmatrix}\mathbf{U}^H\mathbf{U}_s\mathbf{U}_s^H\bar{\mathbf{s}}_0 \\ &= \mathbf{U}\begin{bmatrix} \frac{g_2\lambda_1}{1+g_2\lambda_1} & \cdots & 0 & \mathbf{0} \\ 0 & \ddots & 0 & \vdots \\ 0 & \cdots & \frac{g_2\lambda_K}{1+g_2\lambda_K} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \end{bmatrix}\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &\triangleq \mathbf{U}\mathbf{D}_2\mathbf{U}^H\bar{\mathbf{s}}_0.\end{aligned} \tag{26}$$

The true ASV $\mathbf{s}_0$ is spanned by $\mathbf{U}_s$, i.e.,

$$\mathbf{s}_0 = \mathbf{U}_s\mathbf{c}_0 \tag{27}$$

where $\mathbf{c}_0$ is the coefficient vector. Assume that g_1 is the optimal diagonal loading factor for the RCB. Choosing $g_2 = g_1$, we have

$$\begin{aligned}\mathbf{s}_0^H\mathbf{R}_n^{-1}\mathbf{s}_1 &= \mathbf{c}_0^H\mathbf{U}_s^H\mathbf{R}_n^{-1}\mathbf{U}\mathbf{D}_1\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &= \mathbf{c}_0^H\mathbf{U}_s^H\mathbf{R}_n^{-1}[\mathbf{U}_s\ \mathbf{U}_n]\mathbf{D}_1\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &= \mathbf{c}_0^H[\mathbf{K}\ \mathbf{0}]\mathbf{D}_1\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &= \mathbf{c}_0^H[\mathbf{K}\ \mathbf{0}]\mathbf{D}_2\mathbf{U}^H\bar{\mathbf{s}}_0 \\ &= \mathbf{s}_0^H\mathbf{R}_n^{-1}\mathbf{s}_2,\end{aligned} \tag{28}$$

where $\mathbf{K} = \mathbf{U}_s^H\mathbf{R}_n^{-1}\mathbf{U}_s$.

We have,

$$\begin{aligned}||\mathbf{s}_1||_{\mathbf{R}}^2 &= \bar{\mathbf{s}}_0^H\mathbf{U}\mathbf{D}_1\mathbf{U}^H\mathbf{R}_n^{-1}\mathbf{U}\mathbf{D}_1\mathbf{U}^H\bar{\mathbf{s}}_0 = \mathbf{v}^H\mathbf{D}_1\mathbf{E}\mathbf{D}_1\mathbf{v}, \\ ||\mathbf{s}_2||_{\mathbf{R}}^2 &= \bar{\mathbf{s}}_0^H\mathbf{U}\mathbf{D}_2\mathbf{U}^H\mathbf{R}_n^{-1}\mathbf{U}\mathbf{D}_2\mathbf{U}^H\bar{\mathbf{s}}_0 = \mathbf{v}^H\mathbf{D}_2\mathbf{E}\mathbf{D}_2\mathbf{v}.\end{aligned} \tag{29}$$

where

$$\begin{aligned}\mathbf{E} &= \mathbf{U}^H\mathbf{R}_n^{-1}\mathbf{U}, \\ \mathbf{v} &= \mathbf{U}^H\bar{\mathbf{s}}_0,\end{aligned} \tag{30}$$

The covariance matrix $\mathbf{R}_n$ consists of two parts, $\mathbf{R}_i$ and $\sigma_n^2\mathbf{I}$, which are the covariance matrices of the interferences and background noise, respectively. Since the signal space of the interferences is a subspace of $\mathbf{U}_s$, $\mathbf{R}_n$ can be expressed as

$$\mathbf{R}_n = \mathbf{R}_i + \sigma_n^2\mathbf{I} = [\mathbf{U}_s\ \mathbf{U}_n]\begin{bmatrix}\mathbf{D}_i & \mathbf{0} \\ \mathbf{0} & \sigma_n^2\mathbf{I}\end{bmatrix}[\mathbf{U}_s\ \mathbf{U}_n]^H, \tag{31}$$

where the matrix $\mathbf{D}_i$ of size $K \times K$ is not necessary a diagonal matrix. The inverse matrix of $\mathbf{R}_n^{-1}$ can be expressed as

$$\mathbf{R}_n^{-1} = [\mathbf{U}_s\ \mathbf{U}_n]\begin{bmatrix}\mathbf{D}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^{-2}\mathbf{I}\end{bmatrix}[\mathbf{U}_s\ \mathbf{U}_n]^H. \tag{32}$$

Therefore,

$$\mathbf{E} = \mathbf{U}^H\mathbf{R}_n^{-1}\mathbf{U} = \begin{bmatrix}\mathbf{D}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \sigma_n^{-2}\mathbf{I}\end{bmatrix}, \tag{33}$$

With the derived $\mathbf{E}$ in (33) and new definition of matrix $\mathbf{D}_3$,

$$\mathbf{D}_3 = \begin{bmatrix}0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & \cdots & 0 \\ 0 & \cdots & \frac{g_1\lambda_{K+1}}{1+g_1\lambda_{K+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \frac{g_1\lambda_M}{1+g_1\lambda_M}\end{bmatrix}, \tag{34}$$

we have,

$$\begin{aligned}||\mathbf{s}_1||_{\mathbf{R}}^2 &= \mathbf{v}^H\mathbf{D}_1\mathbf{E}\mathbf{D}_1\mathbf{v} \\ &= \mathbf{v}^H(\mathbf{D}_2+\mathbf{D}_3)\mathbf{E}(\mathbf{D}_2+\mathbf{D}_3)\mathbf{v} \\ &= \mathbf{v}^H\mathbf{D}_2\mathbf{E}\mathbf{D}_2\mathbf{v} + \mathbf{v}^H\mathbf{D}_3\mathbf{E}\mathbf{D}_3\mathbf{v} \\ &= ||\mathbf{s}_2||_{\mathbf{R}}^2 + \mathbf{v}^H\mathbf{D}_3\mathbf{E}\mathbf{D}_3\mathbf{v}\end{aligned} \tag{35}$$

Since $\mathbf{D}_3\mathbf{E}\mathbf{D}_3$ is a non-negative Hermitian matrix, we have

$$||\mathbf{s}_1||_{\mathbf{R}}^2 \geq ||\mathbf{s}_2||_{\mathbf{R}}^2. \tag{36}$$

Denoting the angles between $\mathbf{s}_0$ and $\mathbf{s}_1$ as θ_1 and the angle between $\mathbf{s}_0$ and $\mathbf{s}_2$ as θ_2, we have

$$\cos^2(\theta_1) = \frac{|\mathbf{s}_0^H\mathbf{R}_n^{-1}\mathbf{s}_1|^2}{||\mathbf{s}_0||_{\mathbf{R}}^2||\mathbf{s}_1||_{\mathbf{R}}^2} \leq \frac{|\mathbf{s}_0^H\mathbf{R}_n^{-1}\mathbf{s}_2|^2}{||\mathbf{s}_0||_{\mathbf{R}}^2||\mathbf{s}_2||_{\mathbf{R}}^2} = \cos^2(\theta_2) \tag{37}$$

□

The optimal factor g_2 should be selected to maximize $\cos^2(\theta_2)$, therefore, the corresponding $\cos^2(\theta_2)$ must be greater than or equal to $\cos^2(\theta_1)$.

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