A THEORETICAL ANALYSIS OF 2D SENSOR ARRAYS FOR TDOA BASED LOCALIZATION

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ABSTRACT

In source localization from time difference of arrival, the impact of the sensor array geometry to the localization accuracy is not well understood yet. A first rigorous analysis can be found in [1]. It derived sufficient and necessary conditions for optimum array geometry in terms of minimum Cramer-Rao bound. This paper continues the above work and studies theoretically the localization accuracy of two-dimensional sensor arrays. It addresses different issues: a) optimum vs. uniform angular array b) near-field vs. far-field array c) using all sensor pairs vs. those with a common reference sensor as required from spherical position estimators. The paper ends up with some new insights into the sensor placement problem.

1. INTRODUCTION

The classic applications of source localization from time delay estimates were navigation, surveillance, and aerospace by detecting radio waves (e.g. LORAN and DECCA) [2]. Two emerging applications are positioning of mobile terminals [3, 4] and acoustic localization and tracking by using microphone arrays [5, 6]. Very often the common first step is to estimate the time difference of arrival (TDOA) between the source and a sensor pair [7, 8]. The second step is the source position estimation from the noisy TDOA measurements of different sensor pairs [9, 10, 11, 12, 13, 14].

While much research effort was spent on developing suitable TDOA and position estimators in the last decades, there was no systematic study yet on the impact of the sensor array geometry to the localization accuracy. Most known results are obtained from computer simulations for some example sensor positions. [1] gives a theoretic analysis of the localization accuracy in terms of the Cramer-Rao bound (CRB). It derived sufficient and necessary conditions for an unconstrained optimum sensor placement which minimizes the trace of CRB.

However, many questions are still open:

- In real localization systems, the sensor placement is often subject to geometric constraints because the sensors can be placed only in a spatially restricted region (e.g. far-field). What is the strategy of constrained sensor placement? What is the accuracy loss with respect to the unconstrained optimum solution?
- For an M-element array, one basic assumption in [1] is the use of TDOA estimates of all $M(M-1)/2$ sensor pairs. This is not satisfied for the family of spherical position estimators [11, 13, 14] which all use $M-1$ sensor pairs with a common reference sensor. What is the effect of this smaller number of sensor pairs? Which sensor should be chosen as the reference sensor?
- A very simple sensor placement strategy is to keep equal angular spacing between adjacent sensors. What is the performance loss of this *uniform angular array* (UAA) with respect to the optimum one, for both unconstrained and constrained sensor placement?

This paper gives the answers to the above questions.

2. TDOA BASED SOURCE LOCALIZATION

Assume that the sensor array consists of M sensors at the positions $\underline{q}_i \in \mathbb{R}^D$ $(i = 1, \ldots, M; D = 2, 3)$. The source position vector is $\underline{p} \in \mathbb{R}^D$. The distance between the source and sensor i is $d_i(\underline{p}) = \|\underline{p} - \underline{q}_i\|$. The difference in the distance of the sensors i and j to the source is $d_{ij}(\underline{p}) = d_i(\underline{p}) - d_j(\underline{p})$. Let

$$\tau_{ij} = \frac{1}{v} d_{ij}(\underline{p}) + n_{ij} \tag{1}$$

be the TDOA measured from the signals of sensor i and j. v is the wave propagation speed and n_{ij} is the measurement error. Let I denote the set of N sensor pairs (i, j) whose TDOA estimates are used in localization. By introducing the $N \times 1$ vectors

$$\underline{\tau} = \begin{bmatrix} \tau_{ij} \\ \vdots \end{bmatrix}_{(i,j)\in I}, \quad \underline{d} = \begin{bmatrix} d_{ij} \\ \vdots \end{bmatrix}_{(i,j)\in I}, \quad \underline{n} = \begin{bmatrix} n_{ij} \\ \vdots \end{bmatrix}_{(i,j)\in I}, \tag{2}$$

the signal model becomes

$$\underline{\tau} = \frac{1}{v} \underline{d}(\underline{p}) + \underline{n}. \tag{3}$$

The problem of localization is to estimate the source position vector $\underline{p}$ given $\{\underline{q}_i\}$, $\underline{\tau}$, and v.

Depending on the position estimator used, two different sets of sensor pairs are widely used: The *full TDOA set*

$$I_0 = \{(i, j) | 1 \le i < j \le M\} \tag{4}$$

includes all $M(M-1)/2$ sensor pairs. In contrast, the *spherical TDOA set*

$$I_s = \{(1, k), \ldots, (k-1, k), (k+1, k), \ldots, (M, k)\} \tag{5}$$

contains only those $M-1$ sensor pairs with the common *reference sensor* k. This set is mandatory in all spherical position estimators [11, 13, 14]. We will study both sets in the sequel.

3. CRB AND OPTIMUM ARRAYS

In order to simplify the analysis, we assume that the noise vector $\underline{n}$ in (3) is Gaussian with zero mean and the covariance matrix $\sigma^2\mathbf{I}$. According to [1, 13], the CRB for the source position vector $\underline{p}$ is $\mathbf{J}^{-1} = (v\sigma)^2(\mathbf{G}\mathbf{G}^T)^{-1}$ with

$$\mathbf{G} = [\underline{g}_{ij} \ \ldots]_{(i,j)\in I}, \quad \underline{g}_{ij} = \underline{g}_i - \underline{g}_j, \quad \underline{g}_i = \frac{\underline{p} - \underline{q}_i}{\|\underline{p} - \underline{q}_i\|}. \tag{6}$$

$\underline{g}_i$ is a unit-length vector pointing from sensor i to the source. $\mathbf{G}$ is a $D \times N$ matrix containing the difference vectors $\underline{g}_{ij}$ in columns. As a measure for the localization accuracy, we propose the trace of the CRB

$$f = \mathrm{tr}[\mathbf{J}^{-1}] = (v\sigma)^2 \mathrm{tr}[(\mathbf{G}\mathbf{G}^T)^{-1}]. \tag{7}$$

Clearly, not only the variance σ^2 of TDOA estimates but also the direction vectors $\{\underline{g}_i\}$ affect the localization accuracy. In the literature, the term $\sqrt{\mathrm{tr}[(\mathbf{G}\mathbf{G}^T)^{-1}]}$ which only depends on the array geometry is called the *geometric dilution of precision* (GDOP) [4].

In [1], we have studied how to minimize the GDOP by an optimum sensor placement. It turns out that under the assumption of full TDOA set $I = I_0$,

$$f \geq f_{\min} = (v\sigma)^2 \frac{D^2}{M^2}. \tag{8}$$

The equality holds if and only if $\sum_{i=1}^{M} \underline{g}_i = \underline{0}$ and if the $D \times M$ matrix $\mathbf{g} = [\underline{g}_1 \ \ldots \ \underline{g}_M]$ satisfies $\mathbf{g}\mathbf{g}^T = (M/D)\mathbf{I}$ where $\mathbf{I}$ is the identity matrix. Sensor arrays satisfying these conditions are called *optimum*. One example of two-dimensional ($D = 2$) optimum arrays is given by

$$\underline{g}_i = [\cos\alpha_i,\ \sin\alpha_i]^T, \quad \alpha_i = \alpha_1 + (i-1)\delta, \quad i = 1, \ldots, M \tag{9}$$

with $\delta = 2\pi/M$. More examples can be found in [1]. Below we consider the two-dimensional case only.

4. UNIFORM ANGULAR ARRAY UNDER I_0

Sensor arrays defined by (9) are called *uniform angular array* (UAA) due to the constant angular spacing δ between adjacent sensors. The *angular aperture* of the array is defined as

$$\Lambda = M\delta. \tag{10}$$

Clearly, the previous optimum array is a UAA with the full angular aperture $\Lambda = 2\pi$. The source is surrounded by sensors (near-field) and there are no sensor placement constraints. In far-field problems, however, the source is far away from the sensors (with respect to the sensor spacing) and the angular aperture is much smaller. In this section, we investigate the localization accuracy of UAA with a reduced angular aperture $0 < \Lambda \leq 2\pi$ when using the full TDOA set I_0.

By combining Eqs. (4), (6), (7), and (9), we obtain after some calculations:

$$\begin{aligned} f_0(\Lambda, M) &= f_{0,\min}(M) \cdot L_0(\Lambda, M), \\ L_0(\Lambda, M) &= \frac{1 - \rho_2^2}{(1-\rho_1)(1+\rho_1 - 2\rho_2^2)}, \\ \rho_1 = \rho(\Lambda) &= \frac{\sin(\Lambda)}{M\sin(\Lambda/M)}, \quad \rho_2 = \rho(\Lambda/2). \end{aligned} \tag{11}$$

$f_0(\Lambda, M)$ describes the localization accuracy of an M-element UAA with the angular aperture Λ based on I_0. $f_{0,\min}(M)$ denotes the minimum value of $f_0(\Lambda, M)$ given in (8). Their ratio $L_0(\Lambda, M)$ characterizes the accuracy loss due to a reduced angular aperture. It is called the *loss function*.

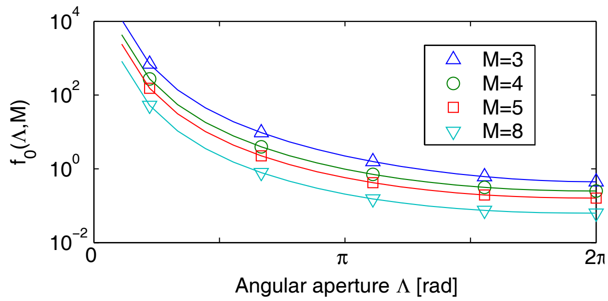

Fig. 1. Localization accuracy of UAA with full TDOA set I_0

Figure 1 shows $f_0(\Lambda, M)$ for $0 < \Lambda \leq 2\pi$ and $M = 3, 4, 5, 8$. Obviously, $f_0(\Lambda, M)$ decreases (the localization accuracy increases) for increasing Λ and M. A function analysis of $f_0(\Lambda, M)$ at different values of Λ and M reveals the following properties:

1. If $\Lambda = 2\pi$, $L_0(2\pi, M) = 1$, i.e. the UAA is optimum.
2. If Λ is close to 2π and $M \gg 1$,
$$L_0(\Lambda, M) \approx 1 + 2\left(1 - \frac{\Lambda}{2\pi}\right)^2. \tag{12}$$
$L_0(\Lambda, M)$ decreases slowly as Λ approaches 2π.
3. If $\Lambda = \pi$,
$$L_0(\pi, M) = \frac{[M\sin(\frac{\pi}{2M})]^2 - 1}{[M\sin(\frac{\pi}{2M})]^2 - 2}. \tag{13}$$
In this case, the loss function decreases monotonously from $L_0(\pi, 3) = 5$ to $L_0(\pi, \infty) \approx 3.1$ for an increasing number of sensors M.
4. If $\Lambda \ll 1$, we obtain the following approximation
$$L_0(\Lambda, M) \approx \frac{180}{(1 - \frac{1}{M^2})(1 - \frac{4}{M^2})\Lambda^4} \overset{M\gg1}{\approx} \frac{180}{\Lambda^4}. \tag{14}$$
The accuracy loss increases dramatically ($\sim 1/\Lambda^4$) for a decreasing angular aperture Λ.
5. If $M \gg 1$, the loss function
$$\begin{aligned} L_0(\Lambda, M) &\approx L_0(\Lambda, \infty) \\ &= \frac{1 - \mathrm{si}^2(\Lambda/2)}{[1 - \mathrm{si}(\Lambda)][1 + \mathrm{si}(\Lambda) - 2\mathrm{si}^2(\Lambda/2)]} \end{aligned} \tag{15}$$

with $\mathrm{si}(x) = \sin(x)/x$ is nearly independent of M. In this case, $f_0(\Lambda, M) \sim 1/M^2$ for all Λ. Table 1 shows some values of $L_0(\Lambda, \infty)$. It helps us to better understand the contribution of the angular aperture to the localization accuracy. Roughly spoken, "the first 90° is much more important than the last 90°". Increasing the angular aperture from $\Lambda = 0$ to $\Lambda = \pi/2$ dramatically improves the accuracy. In contrast, the improvement is vanishing when increasing Λ from $3\pi/2$ to 2π.

Λ	0	$\pi/4$	$\pi/2$	π	$3\pi/2$	2π
$L_0(\Lambda, \infty)$	∞	488.6	33.7	3.1	1.2	1

Table 1. Accuracy loss as a function of the angular aperture Λ when using the full TDOA set I_0

5. UNIFORM ANGULAR ARRAY UNDER I_S

In this section, we perform an accuracy analysis of UAA for the spherical TDOA set I_s in (5). This means, only $M-1$ TDOAs relative to the reference sensor k are used in localization. Due to the reduced number of sensor pairs, we expect a performance loss in terms of the CRB [1]. The question is how much? One additional question concerns the choice of the reference sensor k. Below we first study the case $k = 1$. By combining Eqs. (5), (6), (7), and (9), we obtain after some lengthy calculations:

$$Z = \frac{3-\rho_1^2-2\rho_2^2}{4} - \frac{(\rho_1-\rho_2^2)}{2}\cos\Lambda - \rho_2(1-\rho_1)\cos\frac{\Lambda}{2},$$

$$f_{s,1}(\Lambda, M) = \frac{(v\sigma)^2}{M-1}\cdot\frac{2(1-\rho_2\cos\frac{\Lambda}{2})}{Z}, \qquad (16)$$

$$\rho_1 = \rho(\Lambda) = \frac{\sin(\frac{M-1}{M}\Lambda)}{(M-1)\sin(\frac{1}{M}\Lambda)}, \qquad \rho_2 = \rho(\Lambda/2).$$

$f_{s,k}(\Lambda, M)$ is the trace of CRB based on I_s with the reference sensor k. Figure 2 plots $f_{s,1}(\Lambda, M)$ against Λ for $M = 3, 4, 5, 8$.

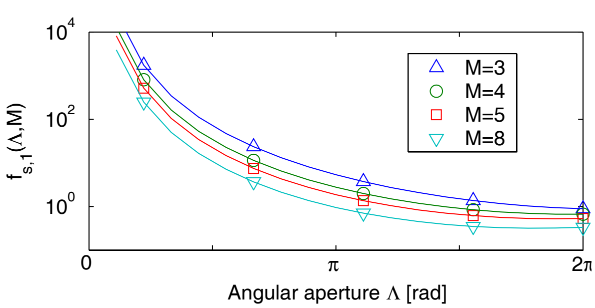

Fig. 2. Localization accuracy of UAA with spherical set I_s

Again, we analyse $f_{s,1}(\Lambda, M)$ in different regions of (Λ, M):

1. At the full angular aperture $\Lambda = 2\pi$,

$$f_{s,1}(2\pi, M) = (v\sigma)^2\frac{8}{3M}. \qquad (17)$$

This value is larger than $f_0(2\pi, M) = (v\sigma)^2(2/M)^2$ in (8) as we used the full TDOA set I_0. In particular, $f_{s,1}(2\pi, M)$ decreases with $1/M$ while $f_0(2\pi, M)$ decreases much faster with $1/M^2$. This is clearly due to the fact that I_s contains $M-1$ while I_0 contains $M(M-1)/2$ TDOA measurements.

2. If $\Lambda = \pi$, we obtain

$$f_{s,1}(\pi, M) = (v\sigma)^2\frac{8(M-1)}{3M^2-4M-4\cot^2(\frac{\pi}{2M})}. \qquad (18)$$

For easy comparison with the full aperture case, we define the ratio

$$r(\Lambda, M) = f_{s,1}(\Lambda, M)/f_{s,1}(2\pi, M). \qquad (19)$$

It turns out that $r(\pi, M)$ decreases monotonously from $r(\pi, 3) = 6$ to $r(\pi, \infty) \approx 2.2$ as M increases. In other words, there is only a moderate improvement when we increase the angular aperture beyond π.

3. If $\Lambda \ll 1$ and $M \gg 1$,

$$f_{s,1}(\Lambda, M) \approx (v\sigma)^2\frac{320}{M\Lambda^4}, \quad r(\Lambda, M) \approx \frac{120}{\Lambda^4}. \qquad (20)$$

For a low aperture UAA, each small increase of Λ results in a huge improvement of the localization accuracy.

4. If $M \gg 1$, ρ_1 and ρ_2 in (16) simplify to $\mathrm{si}(\Lambda)$ and $\mathrm{si}(\Lambda/2)$. They are independent of M. This implies $f_{s,1}(\Lambda, M) \sim 1/M$ for all Λ. Table 2 shows the ratio $r(\Lambda, \infty)$ for some values of Λ. Interestingly, $r(3\pi/2, \infty)$ is smaller than one, indicating that the aperture $3\pi/2$ is even better than the full aperture 2π. This is not surprising because, in contrast to I_0, the full aperture UAA under I_s is not optimum. Optimum arrays for the spherical case I_s are described in the next section.

Λ	0	$\pi/4$	$\pi/2$	π	$3\pi/2$	2π
$r(\Lambda, \infty)$	∞	326.2	22.6	2.2	0.96	1

Table 2. Impact of angular aperture on localization accuracy for the spherical TDOA set I_s

Finally, we study the performance loss of I_s in comparison to I_0 and the effect of the reference sensor k. For this purpose, we consider the ratio of f between I_s and I_0

$$R_k(\Lambda, M) = \frac{f_{s,k}(\Lambda, M)}{f_0(\Lambda, M)}. \qquad (21)$$

For reasons of symmetry, we only consider values of k not larger than $(M+1)/2$. Figure 3 shows $R_k(\Lambda, M)$ for $M = 3, 5, 7, 9$ and different values of k. We observe:

- The performance loss of using I_s instead of I_0 is in the order M. This is mainly because $f_0(\Lambda, M) \sim 1/M^2$ and $f_{s,k}(\Lambda, M) \sim 1/M$.
- This loss has a weak dependence on Λ.

- The choice of the reference sensor k also impacts the CRB. For small aperture values, the middle sensors ($k = \frac{M}{2}$ or $\frac{M+1}{2}$) seem to be optimum. For large aperture, the first sensor ($k = 1$) seems to be the best choice. The boundary between "small' and "large" aperture (the vertical bars in the second row of Figure 3) shifts to the left as M increases. Nevertheless, the overall effect of the choice of the reference sensor is quite limited (at max. factor 2.5 in Fig. 3). It becomes less important for increasing Λ and M.

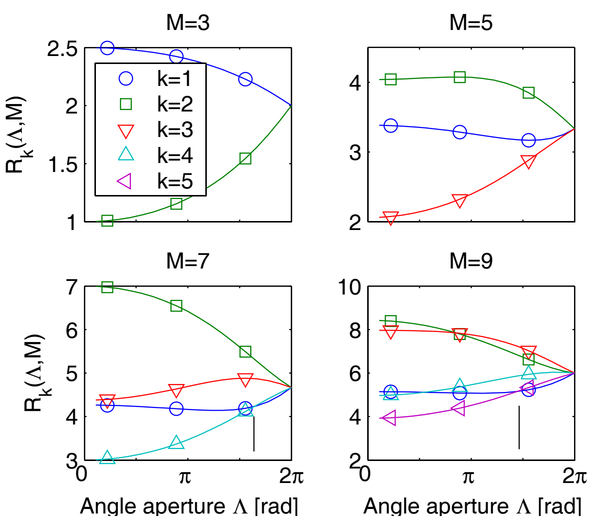

Fig. 3. Accuracy loss of I_s in comparison to I_0

6. OPTIMUM ARRAYS UNDER I_S

The last open question concerns optimum arrays under I_s which minimize the trace of CRB f in (7). Due to limited space, we only present the results without going into details. We assume again that the sensor array geometry is characterized by the M direction vectors $\underline{g}_i = [\cos\alpha_i,\ \sin\alpha_i]^T$. If M is odd, the optimum array consists of one reference sensor at the angle α_0 and two clusters of each $(M-1)/2$ sensors at the angle

$$\alpha_0 \pm \beta \quad \text{with} \quad \beta = 2\arcsin\sqrt{2/3} \approx 109.47^\circ. \tag{22}$$

The corresponding minimum value of f is

$$f_{s,\min} = (\nu\sigma)^2 \frac{27}{16(M-1)}. \tag{23}$$

Note that such a near-field array is realizable because sensors having the same angle with respect to the source may still have different distances to the source. If M is even, the optimum array consists of, in addition to the reference sensor, two clusters of each $M/2$ and $M/2-1$ sensors at the angle $\alpha_0+\beta_1$ and $\alpha_0-\beta_2$ where β_1 and β_2 deviate slightly (depending on M) from the value in (22). Asymptotically, if M approaches infinity, an even-M optimum array also converges to the limit in (22) and (23).

It is interesting to compare the localization accuracy of the full aperture UAA in (17) and the clustered optimum array in (23), both under the spherical TDOA set I_s. Their ratio

$$\frac{f_{s,1}(2\pi, M)}{f_{s,\min}} = \frac{128}{81}\frac{M-1}{M} \approx 1.58 \quad \text{for} \quad M \gg 1 \tag{24}$$

shows only a moderate benefit of the optimum array in comparison to the easily designed full aperture UAA.

7. REFERENCES

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