STATISTICAL RESOLUTION LIMITS OF DOA FOR DISCRETE SOURCES

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ABSTRACT

This paper examines the stochastic Cramer-Rao bound (CRB) of direction of arrival (DOA) estimates for binary phase-shift keying (BPSK), minimum shift keying (MSK) and quaternary phase-shift keying (QPSK) modulated signals in the presence of unknown nonuniform Gaussian noise. After deriving closed-form expressions of the CRB, the statistical resolution limit, defined as the source separation that equals its own CRB is given. It is shown that this highest achievable resolution is proportional to the reciprocal of the fourth root of the product of the number of snaphots by an extended signal to noise ratio (SNR), in contrast to the square root dependence for circular Gaussian sources.

1. INTRODUCTION

There is a considerable literature about array resolution limits (see e.g., [1], and references therein). Based on spectral peaks, various criteria has been first devised to characterize the resolution limits of specific high-resolution algorithms. Then, authors has considered the CRB of the source accuracies themselves (e.g., [2]) or the CRB of the sources separation (e.g., [1]) to define the resolvability of closely spaced sources. But all these works has been devoted to circularly Gaussian distributed signals.

In this contribution, we consider discrete distributed sources such as BPSK, MSK or QPSK modulated signals. First, we extend the closed-form expressions of the stochastic CRB of the DOA alone obtained in [3] for BPSK and QPSK modulations observed on the background of uniform white circular Gaussian noise to BPSK, MSK or QPSK modulations in the presence of nonuniform white noise. Then, considering the statistical resolution limit defined as the source separation that equals its own CRB, we consider the resolvability of BPSK, MSK or QPSK modulated sources compared to those of circular Gaussian sources. Finally, some simulations illustrate the difference of behaviors of these resolution thresholds.

2. DATA MODEL

Consider one or several BPSK, QPSK or MSK modulated signals impinging on an arbitrary array of M sensors. We assume that the array is perfectly calibrated for which the steering vector $(\mathbf{a}_k)$ is a known function of the scalar DOA parameter θ_k, where we suppose $\|\mathbf{a}_k\|^2 = M$. The received signals are bandpass filtered and after down-shifting the sensor signal to baseband, the in-phase and quadrature components are paired to obtain complex signals. We assume Nyquist shaping and ideal sample timing so that the inter-symbol interference at each symbol spaced sampling instance can be ignored. In the absence of frequency offset but with possible phase offset, the signals at the output of the matched filter can be represented as:

$$\mathbf{y}_t = \sum_{k=1}^{K} s_{k,t}\mathbf{a}_k + \mathbf{n}_t \qquad t = 0,\dots,T-1.$$

$s_{k,t} = \sigma_{s_k} e^{i\phi_k}\epsilon_{k,t}$ where $(\epsilon_{k,t})_{t=0,\dots,T-1,k=1,\dots,K}$ are IID random symbols taking values ± 1 [resp. $\pm\sqrt{2}/2 \pm i\sqrt{2}/2$] with equal probabilities for BPSK [resp. QPSK] modulations and for MSK modulations are defined by $\epsilon_{k,t+1} = i\epsilon_{k,t}c_{k,t}$ where $c_{k,t}$ is a sequence of independent BPSK symbols with equal probabilities where the original value $\epsilon_{k,0}$ remains unspecified in the set $\{+1,+i,-1,-i\}$. ϕ_k and σ_{s_k} are considered as unknown parameters. $(\mathbf{n}_t)_{t=1,\dots,T-1}$ are IID M-variate zero-mean complex circular Gaussian random vectors whose covariance matrix is diagonal, parameterized by $\boldsymbol{\sigma} \stackrel{\text{def}}{=} (\sigma_1,\dots,\sigma_M)^T$:

$$\mathbf{Q}_n(\boldsymbol{\sigma}) \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \mathrm{Diag}(\sigma_1^2,\dots,\sigma_M^2).$$

The symbols $\epsilon_{k,t}$ are assumed to be independent from $\mathbf{n}_t$. If no a priori information is available concerning the transmitted symbols, the phases and the different powers, the distribution of $(\mathbf{y}_0,..,\mathbf{y}_{T-1})$ is parameterized by $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\boldsymbol{\sigma},\sigma_{s_1},\phi_1,\theta_1)$ or $\boldsymbol{\alpha} \stackrel{\text{def}}{=} (\boldsymbol{\sigma},\sigma_{s_1},\phi_1,\theta_1,....,\sigma_{s_K},\phi_K,\theta_K)$ according to the number of sources. We note that the MSK modulation is modelled equivalently (see e.g., [4]) by $\epsilon_{k,t} = i^t b_{k,t}\epsilon_{k,0}$ where $b_{k,t}$ is another sequence of independent BPSK symbols $\{-1,+1\}$ with equal probabilities. Consequently, similarly to the BPSK and QPSK modulations, $(\mathbf{y}_t)_{t=0,\dots,T-1}$ are independent M-dimensional random vectors whose probability density function (PDF) of $\mathbf{y}_t$ is mixed circular Gaussian:

$$p(\mathbf{y}_t;\boldsymbol{\alpha}) = \frac{1}{L^K\pi^M \mathrm{Det}(\mathbf{Q}_n(\boldsymbol{\sigma}))} \sum_{j=1}^{L^K} \exp\left(-(\mathbf{y}_t - \mathbf{A}\mathbf{s}_{j,t})^H \mathbf{Q}_n^{-1}(\boldsymbol{\sigma})(\mathbf{y}_t - \mathbf{A}\mathbf{s}_{j,t})\right), \quad (1)$$

with for one source, $K = 1$, $\mathbf{A} = \mathbf{a}_1$ and $\mathbf{s}_{j,t} = \sigma_{s_1}\eta_{j,1}e^{i\phi_1}$, $(\eta_{j,1})_{j=1,L} = \pm 1$ $(L = 2)$, $(\eta_{j,1})_{j=1,L} = \pm\sqrt{2}/2 \pm i\sqrt{2}/2$ $(L = 4)$ or $\mathbf{s}_{j,t} = i^t\sigma_{s_1}\eta_{j,1}e^{i\phi_1}\epsilon_{1,0}$, $(\eta_{j,1})_{j=1,L} = \pm 1$ $(L = 2)$ associated with BPSK, QPSK or MSK modulations, respectively. For example, for two sources, $K = 2$, $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2)$ and $\mathbf{s}_{j,t} = (\sigma_{s_1}\eta_{j,1}e^{i\phi_1}, \sigma_{s_2}\eta_{j,2}e^{i\phi_2})^T$ where $(\eta_{j,1},\eta_{j,2})_{j=1,L^2} = (\pm 1, \pm 1)$ $(L = 2)$, $(\eta_{j,1},\eta_{j,2})_{j=1,L^2} = (\pm\sqrt{2}/2 \pm i\sqrt{2}/2, \pm\sqrt{2}/2 \pm i\sqrt{2}/2)$ $(L = 4)$ or $\mathbf{s}_{j,t} = (i^t\sigma_{s_1}\eta_{j,1}e^{i\phi_1}\epsilon_{1,0}, i^t\sigma_{s_2}\eta_{j,2}e^{i\phi_2}\epsilon_{2,0})^T$, $(\eta_{j,1},\eta_{j,2})_{j=1,L^2} = (\pm 1, \pm 1)$ $(L = 2)$ associated with BPSK, QPSK or MSK modulations, respectively.

3. STOCHASTIC CRB FOR BPSK, QPSK AND MSK SOURCES

3.1. Single source case

Using the whitening transform $\tilde{\mathbf{y}}_t \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}(\boldsymbol{\sigma})\mathbf{y}_t$ and $\tilde{\mathbf{a}}_1 \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}(\boldsymbol{\sigma})\mathbf{a}_1$, the PDF's (1) take the following forms:

$$p(\mathbf{y}_t;\boldsymbol{\alpha}) = \frac{1}{\pi^M \mathrm{Det}(\mathbf{Q}_n(\boldsymbol{\sigma}))} e^{-(\|\tilde{\mathbf{y}}_t\|^2 + \sigma_{s_1}^2\|\tilde{\mathbf{a}}_1\|^2)} c(\tilde{\mathbf{y}}_t)$$

where

$$\begin{aligned} c(\tilde{\mathbf{y}}_t) &= \cosh\left(\sigma_{s_1} g_1(\tilde{\mathbf{y}}_t)\right) \\ &= \cosh\left(\frac{\sigma_{s_1}}{\sqrt{2}} g_1(\tilde{\mathbf{y}}_t)\right)\cosh\left(\frac{\sigma_{s_1}}{\sqrt{2}} g_2(\tilde{\mathbf{y}}_t)\right) \\ &= \cosh\left(\sigma_{s_1} g_3(\tilde{\mathbf{y}}_t)\right) \end{aligned}$$

for the BPSK, QPSK and MSK modulations respectively, with $g_1(\tilde{\mathbf{y}}_t) \stackrel{\text{def}}{=} 2\Re(e^{i\phi_1}\tilde{\mathbf{y}}_t^H\tilde{\mathbf{a}}_1)$, $g_2(\tilde{\mathbf{y}}_t) \stackrel{\text{def}}{=} 2\Im(e^{i\phi_1}\tilde{\mathbf{y}}_t^H\tilde{\mathbf{a}}_1)$ and $g_3(\tilde{\mathbf{y}}_t) \stackrel{\text{def}}{=} 2\Re(i^t e^{i\phi_1}\epsilon_{1,0}\tilde{\mathbf{y}}_t^H\tilde{\mathbf{a}}_1)$. Noting that the structure of these PDF's is similar to those obtained for BPSK and QPSK modulations with uniform white noise in [3] and that $\|\tilde{\mathbf{a}}_1\|^2 = \sum_{m=1}^M \frac{1}{\sigma_m^2}$ independent of θ_1 which is crucial for the proof in [3], the derivations of [3] extend, and we prove in [5] the following theorem:

Theorem 1 *The power $(\boldsymbol{\sigma}, \sigma_{s_1})$ and phase (ϕ_1, θ_1) parameters are decoupled in the Fisher information matrices (FIM) associated with the BPSK, QPSK and MSK modulations:*

$$\begin{aligned} \mathbf{I}_{\mathrm{BPSK}} &= \mathbf{I}_{\mathrm{MSK}} = T\begin{bmatrix} \mathbf{I}_{\mathrm{B}}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\mathrm{B}}^{(2)} \end{bmatrix} \\ \mathbf{I}_{\mathrm{QPSK}} &= T\begin{bmatrix} \mathbf{I}_{\mathrm{Q}}^{(1)} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_{\mathrm{Q}}^{(2)} \end{bmatrix} \end{aligned}$$

with

$$\mathbf{I}_{\mathrm{B}}^{(2)} = \begin{bmatrix} 2\sigma_{s_1}^2\|\tilde{\mathbf{a}}_1\|^2(1 - f(Mr_1)) & 2\sigma_{s_1}^2 i(\tilde{\mathbf{a}}_1'^H\tilde{\mathbf{a}}_1)(1 - f(Mr_1)) \\ 2\sigma_{s_1}^2 i(\tilde{\mathbf{a}}_1'^H\tilde{\mathbf{a}}_1)(1 - f(Mr_1)) & 2\sigma_{s_1}^2\|\tilde{\mathbf{a}}_1'\|^2(1 - f(Mr_1)) \end{bmatrix}$$

$$(\mathbf{I}_{\mathrm{Q}}^{(2)})_{1,1} = 2\sigma_{s_1}^2\|\tilde{\mathbf{a}}_1\|^2\left(1 - (1 + Mr_1)f(\frac{Mr_1}{2})\right)$$

$$(\mathbf{I}_{\mathrm{Q}}^{(2)})_{1,2} = 2\sigma_{s_1}^2 i(\tilde{\mathbf{a}}_1'^H\tilde{\mathbf{a}}_1')\left(1 - (1 + Mr_1)f(\frac{Mr_1}{2})\right)$$

$$(\mathbf{I}_{\mathrm{Q}}^{(2)})_{2,2} = 2\sigma_{s_1}^2\|\tilde{\mathbf{a}}_1'\|^2\left(1 - (1 + \frac{Mr_1}{\|\tilde{\mathbf{a}}_1\|^2}\frac{|\tilde{\mathbf{a}}_1^H\tilde{\mathbf{a}}_1'|^2}{\|\tilde{\mathbf{a}}_1'\|^2})f(\frac{Mr_1}{2})\right)$$

where the SNR is defined as in [6, rel. (48)] by $r_1 \stackrel{\text{def}}{=} \frac{\sigma_{s_1}^2}{M}\sum_{m=1}^M \frac{1}{\sigma_m^2}$, $\mathbf{a}_1' \stackrel{\text{def}}{=} \frac{d\mathbf{a}_1}{d\theta_1}$, $\tilde{\mathbf{a}}_1' \stackrel{\text{def}}{=} \mathbf{Q}_n^{-1/2}(\boldsymbol{\sigma})\mathbf{a}_1'$ and where $f(\rho)$ is the following decreasing function of ρ:

$$f(\rho) \stackrel{\text{def}}{=} \frac{e^{-\rho}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{-\frac{u^2}{2}}}{\cosh(u\sqrt{2\rho})}du.$$

Because these FIM are block diagonal, the following explicit expressions for the CRB for the parameter DOA alone are easily derived:

$$\begin{aligned} \mathrm{CRB}_{\mathrm{BPSK}}(\theta_1) &= \mathrm{CRB}_{\mathrm{MSK}}(\theta_1) \\ &= \frac{1}{T}\left(\frac{1}{\alpha_1 r_1}\right)\left(\frac{1}{1 - f(Mr_1)}\right) \quad (2) \\ \mathrm{CRB}_{\mathrm{QPSK}}(\theta_1) &= \frac{1}{T}\left(\frac{1}{\alpha_1 r_1}\right)\left(\frac{1}{1 - f(\frac{Mr_1}{2})}\right) \quad (3) \end{aligned}$$

where α_1 is the noise dependant factor $2M\left(\sum_{m=1}^M \frac{1}{\sigma_m^2}\right)^{-1}\tilde{\mathbf{a}}_1'^H\mathbf{\Pi}_{\tilde{\mathbf{a}}_1}^{\perp}\tilde{\mathbf{a}}_1'$. We note that

$$\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1) = \mathrm{CRB}_{\mathrm{MSK}}(\theta_1) < \mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)$$

and that for $(\sigma_m^2)_{m=1,\dots,M} = \sigma_n^2$, $\alpha_1 = 2\mathbf{a}_1'^H\mathbf{\Pi}_{\mathbf{a}_1}^{\perp}\mathbf{a}_1'$ and (2) and (3) give the expressions in [3]. The a priori information of uniform white noise does not improve the performance of estimation of the DOA. Finally, note that the whitening approach we use, does not allow us to extend these results to the case of general parameterized noise fields because the crucial property "$\|\tilde{\mathbf{a}}_1\|^2$ does not depend on θ_1" vanishes.

3.2. Several sources case

We consider now two independent BPSK, MSK or QPSK distributed sources (for simplicity of notations but the following results apply for an arbitrary number of sources). Because the PDF of $\mathbf{y}_t$ is a mixture of 4 or 16 Gaussian PDFs, the FIMs associated with the parameter $(\boldsymbol{\sigma}, \sigma_{s_1}, \phi_1, \theta_1, \sigma_{s_2}, \phi_2, \theta_2)$ appears to be prohibitive to compute. The same difficulty occurred in the case of white uniform noise in [3]. However at high SNR's (more precisely for $\sum_{m=1}^M \frac{\sigma_{s_1}^2}{\sigma_m^2} \gg 1$ and $\sum_{m=1}^M \frac{\sigma_{s_2}^2}{\sigma_m^2} \gg 1$, the derivations of [3] extend and we prove in [5] that the FIM's associated with BPSK, QPSK and MSK modulations are approximated by the same following explicit expression:

$$\mathbf{I}_{\mathrm{BPSK}}^{\mathrm{HSNR}} \approx \mathbf{I}_{\mathrm{QPSK}}^{\mathrm{HSNR}} \approx \mathbf{I}_{\mathrm{MSK}}^{\mathrm{HSNR}} \approx T\begin{bmatrix} \mathbf{I}' & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_1' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_2' \end{bmatrix} \quad (4)$$

with $\mathbf{I}' = \mathrm{Diag}(\frac{1}{\sigma_1^2}, \ldots, \frac{1}{\sigma_M^2})$

$$\mathbf{I}'_k = \begin{bmatrix} 2\sum_{m=1}^{M} \frac{1}{\sigma_m^2} & 0 & 0 \\ 0 & 2\sum_{m=1}^{M} \frac{\sigma_{s_k}^2}{\sigma_m^2} & 2\sigma_{s_k}^2(i\tilde{\mathbf{a}}_k'^H \tilde{\mathbf{a}}_k) \\ 0 & 2\sigma_{s_k}^2(i\tilde{\mathbf{a}}_k'^H \tilde{\mathbf{a}}_k) & 2\sigma_{s_k}^2 \|\tilde{\mathbf{a}}_k'\|^2 \end{bmatrix},$$

$k = 1, 2$. We clearly see that the entries corresponding to sources 1 and 2 and noise are decoupled. Consequently, for large SNR's and independent sources, *the CRB for the DOA of one source is independent of the parameters of the other source* and we obtain for the parameter alone $\boldsymbol{\theta} = (\theta_1, \theta_2)$:

$$\begin{aligned} \mathrm{CRB}_{\mathrm{BPSK}}^{\mathrm{HSNR}}(\boldsymbol{\theta}) &\approx \mathrm{CRB}_{\mathrm{QPSK}}^{\mathrm{HSNR}}(\boldsymbol{\theta}) \approx \mathrm{CRB}_{\mathrm{MSK}}^{\mathrm{HSNR}}(\boldsymbol{\theta}) \\ &\approx \frac{1}{T}\begin{bmatrix} \frac{1}{\alpha_1 r_1} & 0 \\ 0 & \frac{1}{\alpha_2 r_2} \end{bmatrix} \end{aligned} \quad (5)$$

where α_2 and r_2 are defined as α_1 and r_1. Furthermore, these CRB's for each DOA are those of the single source case. We note that this property is quite different from the behavior of the CRB under the circular Gaussian distribution and the deterministic CRB, for which the CRB for the DOA of one source depends on the DOA separation [7]. More precisely, it is proved in [5]:

Theorem 2 *The CRB under the circular Gaussian distribution and the deterministic CRB are equivalent (i.e., their ratio tends to 1) as the SNR σ^2/σ_n^2 tend to $+\infty$ where $\mathbf{Q}_n = \sigma_n^2 \mathbf{Q}'_n$ and $\mathbf{R}_n = \sigma^2 \mathbf{R}'_n$ with $\mathbf{Q}'_n$ and $\mathbf{R}'_n$ diagonal fixed.*

$$\begin{aligned} \mathrm{CRB}_{\mathrm{CG}}(\boldsymbol{\theta}) &\sim \mathrm{CRB}_{\mathrm{DET}}(\boldsymbol{\theta}) \\ &= \frac{1}{T}\begin{bmatrix} \frac{1}{\beta_1(\theta_1,\theta_2)\sigma_{s_1}^2} & 0 \\ 0 & \frac{1}{\beta_2(\theta_1,\theta_2)\sigma_{s_2}^2} \end{bmatrix} \end{aligned} \quad (6)$$

where $\beta_k(\theta_1, \theta_2)$ is the noise dependent factor $2\tilde{\mathbf{a}}_k'^H \mathbf{\Pi}_{\tilde{\mathbf{A}}}^{\perp} \tilde{\mathbf{a}}'_k$, $k = 1, 2$, and $\tilde{\mathbf{A}} \stackrel{\mathrm{def}}{=} (\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2)$. This extends the previous result proved in [7, result R9] in the uniform white noise case for which, $\mathbf{Q}'_n = \mathbf{I}_M$ and $\beta_k(\theta_1, \theta_2) \stackrel{\mathrm{def}}{=} 2\mathbf{a}_k'^H \mathbf{\Pi}_{\mathbf{A}}^{\perp} \mathbf{a}'_k$.

In the case of arbitrary SNR's where no closed-form expressions of the CRB are available, we use a numerical approximation derived from the strong law of large numbers, i.e., $\mathrm{CRB}_{(\mathrm{PSK})}(\boldsymbol{\theta}) = \left(\mathbf{I}_{\mathrm{PSK}}^{-1}\right)_{([M+3\ M+6],[M+3\ M+6])}$ where the FIM associated with $(\mathbf{y}_0, .., \mathbf{y}_{T-1})$ is given by: $(\mathbf{I}_{\mathrm{PSK}})_{k,l} = \lim_{T'\to\infty} \frac{1}{T'} \sum_{t'=0}^{T'-1} \sum_{t=0}^{T-1} \left(\frac{\partial \ln p(\mathbf{y}_{t+t'T};\boldsymbol{\alpha})}{\partial \alpha_k}\right) \left(\frac{\partial \ln p(\mathbf{y}_{t+t'T};\boldsymbol{\alpha})}{\partial \alpha_l}\right)$.
These expressions allow us to specify the domain of validity of the high SNR approximation (4).

4. STATISTICAL RESOLUTION LIMIT

Despite the CRB does not directly indicate the best resolution achievable by an unbiased estimator, it can be used to define an absolute limit of resolution. Following the criterium described in [1], two sources are meaningfully resolved if the root mean square of the CRB of the estimated DOA separation $(\theta_{1,T} - \theta_{2,T})$ is less than the DOA separation $\Delta\theta \stackrel{\mathrm{def}}{=} |\theta_1 - \theta_2|$.

$$\sqrt{\mathrm{CRB}_{\mathrm{PSK}}(\theta_1 - \theta_2)} < \Delta\theta. \quad (7)$$

As a consequence of the emphasized difference of behavior of the CRB for large SNR's for discrete sources compared to circular Gaussian sources, the behavior of the resolution threshold for two closely spaced independent sources is also quite different.

4.1. Discrete sources

Applied to discrete sources, criterium (7) gives:

$$\sqrt{\frac{1}{T}\left(\frac{1}{\alpha_1 r_1} + \frac{1}{\alpha_2 r_2}\right)} < \Delta\theta$$

because θ_1 and θ_2 are decoupled in (5). With $\frac{1}{r_e} \stackrel{\mathrm{def}}{=} \frac{1}{\alpha_1 r_1} + \frac{1}{\alpha_2 r_2}$ where r_e can be interpreted as an extended SNR[1], we obtain the following statistical resolution limit:

$$\Delta\theta > \frac{1}{(T r_e)^{1/2}} \quad (8)$$

which compared to Gaussian distributed sources in the next subsection is quite different.

4.2. Circular Gaussian sources

Noting that deterministic CRB (6) has a similar form in the uniform and nonuniform white noise case, providing the whitening steering vectors $\tilde{\mathbf{a}}_k$ are considered, the approach used in [2] for two closely spaced sources in the uniform case can be applied as well in the nonuniform case. Following the second order Taylor expansion in $\Delta\theta$ of [2, (31)] with $\theta_k = \theta_0 \pm \frac{\Delta\theta}{2}$ with $\theta_0 \stackrel{\mathrm{def}}{=} \frac{\theta_1+\theta_2}{2}$, we obtain the equivalence[2]: $\beta_k(\theta_1, \theta_2) \sim 2\|\tilde{\boldsymbol{\epsilon}}\|^2(\Delta\theta)^2$ with $\tilde{\boldsymbol{\epsilon}} = \mathbf{\Pi}_{\tilde{\dot{\mathbf{A}}}}^{\perp} \tilde{\dot{\mathbf{a}}}_0^{"}$ where $\tilde{\dot{\mathbf{A}}} \stackrel{\mathrm{def}}{=} \mathbf{Q}_n^{-1/2} \dot{\mathbf{A}}$, $\tilde{\dot{\mathbf{a}}}_0^{"} \stackrel{\mathrm{def}}{=} \mathbf{Q}_n^{-1/2} \mathbf{a}_0^{"}$, $\mathbf{a}_0^{"} \stackrel{\mathrm{def}}{=} \frac{d^2\mathbf{a}_0}{d\theta_0^2}$ and $\dot{\mathbf{A}} \stackrel{\mathrm{def}}{=} [\mathbf{a}_0, \mathbf{a}'_0]$ with $\mathbf{a}_0$ is the steering vector associated with θ_0. Consequently with $\frac{1}{r'_e} \stackrel{\mathrm{def}}{=} \frac{1}{2\|\tilde{\boldsymbol{\epsilon}}\|^2}(\frac{1}{\sigma_{s_1}^2} + \frac{1}{\sigma_{s_2}^2})$ which defines another extended SNR[3] r'_e, we obtain the following statistical resolution limit:

$$\Delta\theta > \frac{1}{(T r'_e)^{1/4}} \quad (9)$$

[1] In the case of a uniform linear array (ULA), $\alpha_k r_k = 2\sigma_{s_k}^2 [\sum_{m=1}^{M-1} \frac{m^2}{\sigma_m^2} - (\sum_{m=1}^{M-1} \frac{m}{\sigma_m^2})^2 (\sum_{m=1}^{M} \frac{1}{\sigma_m^2})^{-1}]$ which gives for uniform white noise, $\alpha_1 = \alpha_2 = M(M^2-1)/6$ and consequently $r_e = \frac{M(M^2-1)}{6} \frac{\sigma_{s_1}^2 \sigma_{s_2}^2}{\sigma_n^2(\sigma_{s_1}^2 + \sigma_{s_2}^2)}$.

[2] The ratio of the two terms tends to 1 when $\Delta\theta$ tends to 0.

[3] In the case of uniform white noise for a uniform linear array $r'_e = \frac{(M-1)(9M^3-3M^2-6M+2)}{30M} \frac{\sigma_{s_1}^2 \sigma_{s_2}^2}{\sigma_n^2(\sigma_{s_1}^2+\sigma_{s_2}^2)}$.

5. ILLUSTRATIVE EXAMPLES

The purpose of this section is to illustrate the large difference of behavior of the resolution limit obtained for discrete or circular Gaussian distributed sources. We consider throughout this section two independent equipowered sources impinging on a ULA of $M = 6$ or 10 sensors separated by a half-wavelength for which $\mathbf{a}_k = (1, e^{i\theta_k}, ..., e^{i(M-1)\theta_k})^T$ with the nonuniform [resp. uniform] white noise model given in [6] for the first two [resp. the third] figures.

Fig.1 compares $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ with the CRB under the circular complex Gaussian distribution $\mathrm{CRB}_{\mathrm{CG}}(\theta_1)$ under the same a priori that the two sources are independent. We see that $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ is much smaller than $\mathrm{CRB}_{\mathrm{CG}}(\theta_1)$ for large values of DOA separations and SNR's included. Consequently, the ML estimators that take these discrete distributions into account outperform the stochastic ML estimator under the circular Gaussian distribution and the weighed subspace fitting estimator which both reach $\mathrm{CRB}_{\mathrm{CG}}(\theta_1)$. Consequently, the EM approaches [8] that are iterative procedures capable of implementing the stochastic ML estimator under these discrete distributions outperform the ML estimator under circular Gaussian distribution.

Fig.2 exhibits the domain of validity of the high SNR approximation. We see from this figure that this domain depends not only on M, SNR and DOA separation, but also on the distributed sources. It is shown that this domain reduces for QPSK sources compared to BPSK and MSK sources (which have the same domain of validity). The larger the DOA separation is or the larger M is, the larger the domain of validity of the approximation is.

Finally Fig.3 exhibits $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and the estimated mean square error (MSE) $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm initialized by the estimate given by the MUSIC-like algorithm described in [9], as a function of the DOA separation for two SNR's. We see that contrary to $\mathrm{CRB}_{\mathrm{CG}}(\theta_1)$, $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ does not increase significantly when decreasing the DOA separation.

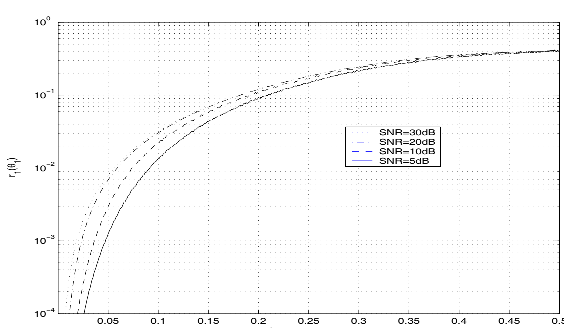

Fig.1 Ratio $r_1(\theta_1) \overset{\mathrm{def}}{=} \frac{\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)}{\mathrm{CRB}_{\mathrm{CG}}(\theta_1)}$ as a function of the DOA separation for different values of SNR's.

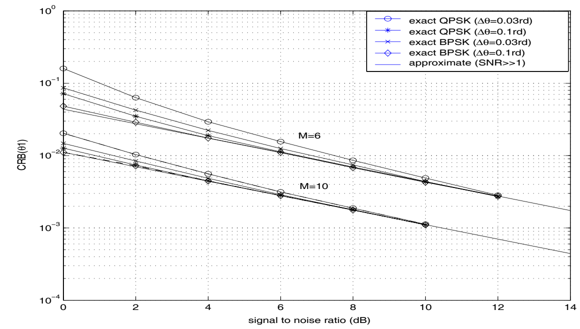

Fig.2 Approximate and exact value (obtained thanks to the strong law of large numbers with T'=10000) of $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and $\mathrm{CRB}_{\mathrm{QPSK}}(\theta_1)$ as a function of the SNR for different values of the DOA separation.

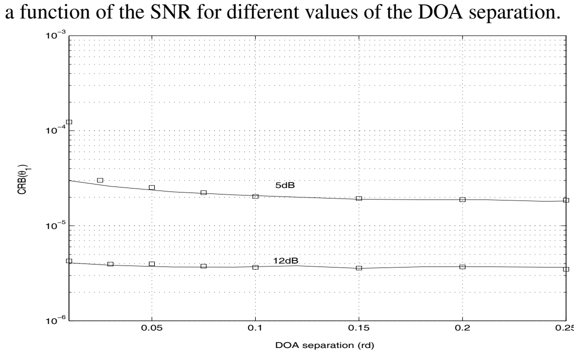

Fig.3 $\mathrm{CRB}_{\mathrm{BPSK}}(\theta_1)$ and estimated MSE $\mathrm{E}(\theta_{1,T} - \theta_1)^2$ given by the deterministic EM algorithm (10 iterations) as a function of the DOA separation for $\Delta\phi = 0.1$rd.

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