ACCURATE APPROXIMATION OF ERROR PROBABILITY ON MIMO CHANNELS AND ITS APPLICATION TO ADAPTIVE MODULATION AND ANTENNA SELECTION

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ABSTRACT

A new approximation for the conditional error probability on quasi-static multiple antenna (MIMO) channels is proposed. For a fixed channel matrix, it is possible to predict the performance of quadrature-amplitude modulations (QAM) transmitted over the MIMO channel in presence of additive white Gaussian noise (AWGN). The tight approximation is based on a simple union bound for the point error probability in the n-dimensional real space. A Pohst or a Schnorr-Euchner lattice enumeration is used to limit the local Theta series inside a finite radius sphere. As applications to this approximation, we describe a new adaptive QAM modulation and a new antenna selection criterion.

1. INTRODUCTION

The achievable information rate of conventional systems with a single antenna at both transmitter and receiver is limited by the modulation size. Therefore, most of recent wireless systems use multiple transmit and receive antennas to achieve higher data rates with a high diversity order [9]. Several techniques have been proposed to improve the performance of these multiple antenna systems regarding the wireless channel conditions, such as adaptive modulation [6][7] and antenna selection [5]. In both cases, to select the appropriate modulation or antenna set to be considered at the transmission, a relevant metric has to be considered to precisely assess the MIMO scheme performance. Taricco and Biglieri gave the exact pairwise error probability in [8] for frequency non-selective multiple antenna systems. The pairwise error probability considered in their paper is the mathematical expectation over all channel realizations. Thus, their closed form expression cannot be used for adaptive modulation nor for antenna selection. In this paper, we propose a new approximation of the conditional error probability in a MIMO system. Based on this tight approximation conditioned on a fixed channel realization, two examples of application are described in this article: adaptive modulation and antenna selection. The paper is organized as follows. Section 2 introduces the notations and the channel model. The error probability approximation is given in section 3. Sections 4 and 5 describe respectively two applications proposed for this approximation: adaptive modulation and antenna selection. Conclusions are drawn in the last section.

2. SYSTEM MODEL

We consider an uncoded transmission system with n_t transmit (Tx) antennas and n_r receive (Rx) antennas. The channel is assumed to be frequency non-selective and quasi-static. The $n_t \times n_r$ MIMO channel matrix $\mathbf{H} = [h_{i,j}]$ is constant during T_c channel uses, where T_c is the channel coherence time. In the latter, one time unit is equal to one transmission period. The coefficients $h_{i,j}$ are independent zero-mean unit-variance complex Gaussian variables that take independent values each T_c periods. For one channel use, the input-output model is

$$\mathbf{r} = \mathbf{s}\mathbf{H} + \boldsymbol{\nu} \quad (1)$$

where $\mathbf{s} = (s_1, \ldots, s_{n_t})$ is the transmit complex vector, $\mathbf{r}$ is the receive vector and $\boldsymbol{\nu}$ is an AWGN with variance N_0 per real dimension. Each Tx symbol s_k belongs to a M_k-QAM, $\Re(s_k)$ and $\Im(s_k) \in \{\pm 1, \pm 3, \ldots, \pm(\sqrt{M_k}-1)\}$, $k = 1 \ldots n_t$, $\Re()$ and $\Im()$ stand for real and imaginary parts. The n_t QAM constellations are not necessary identical, their Cartesian product is denoted C_{QAM}. Since $\mathbf{s}$ is limited to $C_{\text{QAM}} \subset \mathbb{Z}^{2n_t}$, then $\mathbf{x} = \mathbf{s}\mathbf{H}$ belongs to a finite set denoted $C^{\mathbf{H}}$, called receive constellation. The cardinality of the constellation $C^{\mathbf{H}}$ is $\prod_{k=1}^{n_t} M_k$. The spectral efficiency of the uncoded QAM system is $\sum_{k=1}^{n_t} \log_2(M_k)$ bits per channel use. It is assumed that perfect channel state information (CSI) is available only at the receiver . The detection is based on Maximum-Likelihood (ML) criterion. thanks to e.g. a sphere decoder [1][10] if $n_t \leq n_r$ or a rank deficient MIMO system sphere decoder [4] for $n_t > n_r$.

3. ERROR PROBABILITY APPROXIMATION

3.1. Error probability approximation for $n_t \leq n_r$

Without loss of generality we assume that $n_t = n_r$. The study is similar for the asymmetric channel, when $n_r > n_t$.

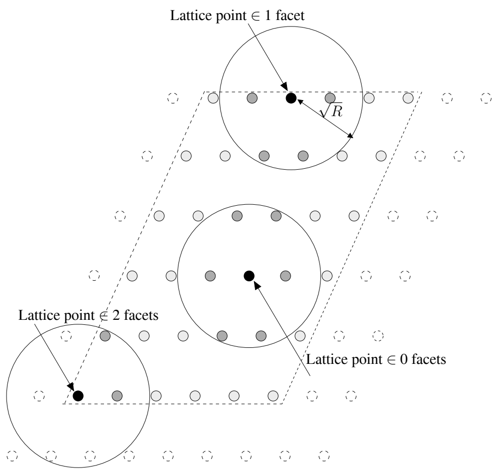

Fig. 1. An example of lattice constellation in $\mathbb{R}^2$. Points are distinguished according to the number of crossing facets.

The performance study of the quasi-static multiple antenna model given in (1) is carried out thanks to lattices and sphere packings theory [2]. The product $\mathbf{x} = \mathbf{sH}$ can be interpreted as a point in the Euclidean space $\mathbb{R}^n$, $n = 2n_t = 2n_r$. The point $\mathbf{x} \in C^{\mathbf{H}}$ belongs to a real lattice Λ of rank n. This means that $C^{\mathbf{H}}$ is a finite subset of the lattice Λ. The real version of $\mathbf{H}$ corresponds to a generator block matrix $\mathbf{G} = [g_{ij}] = [G_{ij}]$, where the block G_{ij} is

$$\begin{bmatrix} g_{2i,2j} & g_{2i,2j+1} \\ g_{2i+1,2j} & g_{2i+1,2j+1} \end{bmatrix} = \begin{bmatrix} \Re(h_{i,j}) & \Im(h_{i,j}) \\ -\Im(h_{i,j}) & \Re(h_{i,j}) \end{bmatrix}. \quad (2)$$

The lattice representation of a multiple antenna channel converts the MIMO model given in (1) into a simple AWGN channel model $\mathbf{r} = \mathbf{x} + \boldsymbol{\nu}$. It is noticed that for a given constellation point $\mathbf{x} = \mathbf{sH}$, the local Theta series [2] (*i.e.* the number of points surrounding $\mathbf{x}$ at a distance d) depends on the position of $\mathbf{x}$ within the constellation. Given this observation, the constellation $C^{\mathbf{H}}$ can be partitioned into $n+1$ subsets

$$C^{\mathbf{H}} = \bigcup_{\ell=0}^{n} I_\ell \quad (3)$$

where I_ℓ contains lattice points located on the intersection of ℓ facets in $C^{\mathbf{H}}$. To illustrate the above observation, we consider an example of lattice constellation within $\mathbb{R}^2$ as depicted on Fig. 1. The local Theta series of the indicated points (black filled) are not identical.

Considering (3), the error probability of the constellation can be written as

$$Pe(C^{\mathbf{H}}) = \sum_{\ell=0}^{n} p_\ell Pe(I_\ell) \quad (4)$$

where the factor p_ℓ is the probability that a point of the constellation $C^{\mathbf{H}}$ belongs to the subset I_ℓ and $Pe(I_\ell)$ is the error probability associated to I_ℓ. When the same M-QAM is applied on all Tx antennas, p_ℓ is expressed, using binomial law of parameter $2/\sqrt{M}$ (probability to be at the edge), as

$$p_\ell = \begin{pmatrix} n \\ \ell \end{pmatrix} \left(\frac{2}{\sqrt{M}}\right)^{\ell} \left(1 - \frac{2}{\sqrt{M}}\right)^{n-\ell}. \quad (5)$$

When distinct QAMs are used, p_ℓ is given by

$$p_\ell = \sum_{L_{\ell,j}} \prod_{i=1}^{n} \left(\frac{2}{\sqrt{M_{[(i+1)/2]}}}\right)^{\ell_i^j} \left(1 - \frac{2}{\sqrt{M_{[(i+1)/2]}}}\right)^{1-\ell_i^j} \quad (6)$$

where $L_{\ell,j} = (\ell_1^j \ldots \ell_i^j \ldots \ell_n^j)$ denotes a length n binary vector whose components satisfy the sum condition $\ell = \sum_{i=1}^{n} \ell_i^j$, $\ell \in [0 \ldots n]$ and $1 \leq j \leq \begin{pmatrix} n \\ \ell \end{pmatrix}$. The integer ℓ_i^j is set to 1 if $\Re(s_i)$ or $\Im(s_i)$ belong to $\{\pm(\sqrt{M_{\lceil i/2 \rceil}} - 1)\}$.
Applying the union upper bound on $Pe(I_\ell)$, the error probability $Pe(C^{\mathbf{H}})$ is upper bounded by

$$Pe(C^{\mathbf{H}}) \leq \sum_{\ell=0}^{n} p_\ell \frac{1}{|I_\ell|} \sum_{\mathbf{x} \in I_\ell} \sum_{i=1}^{i_{\max}} \tau_{\mathbf{x},\ell,i} \times \mathrm{Q}\left(\frac{d_i}{2\sqrt{N_0}}\right) \quad (7)$$

where $\tau_{\mathbf{x},\ell,i}$ is the number of points located at a distance d_i from the point $\mathbf{x} \in I_\ell$. The coefficients $\tau_{\mathbf{x},\ell,i}$ of the local Theta series are easily determined from the original Theta series of the random lattice Λ as follows:
Step1: Generate lattice points $\mathbf{y} = \mathbf{zG} \in \Lambda$ located at a distance d_i from the origin, $\mathbf{z} \in \mathbb{Z}^n$. The Gram matrix $\mathbf{M} = \mathbf{GG}^T$ is full rank and positive-definite, so the *Short Vectors* algorithm could be applied to find these points [3], with polynomial complexity in n.
Step2: For each $\mathbf{y}$ found in the previous step, check if the translate $\mathbf{y} + \mathbf{x}$ belongs to the constellation $C^{\mathbf{H}}$ and increment $\tau_{\mathbf{x},\ell,i}$ accordingly.
To compute numerical results, we limited the number of points in (7) to $N_x = \min(1000, \prod_{k=1}^{n_t} M_k)$. The size of a subset I_ℓ is approximated by $|I_\ell| \approx p_\ell \times N_x$. The number of shells in the local Theta series has been limited to a number i_{max} where the most distant shell is at $2d_{Emin}^2(\Lambda)$, where d_{Emin} is the minimum Euclidean distance of the lattice. The conventional factor 2 is fully justified by its corresponding 3dB signal-to-noise ratio attenuation. If the local Theta series (around $\mathbf{x}$) is empty, then the new search radius can be increased up to $4d_{Emin}^2(\Lambda)$ (6dB attenuation).

3.2. Error probability approximation for $n_t > n_r$

In this subsection, we discuss the case when $n_t > n_r$, i.e., when less observations are available at the receiver than the number of independent transmit data streams. The expressions (5) to (7) remain valid for the studied case. The $n \times n$ matrix $\mathbf{M} = \mathbf{GG}^T$ is rank deficient and the evaluation of the coefficients $\tau_{\mathbf{x},\ell,i}$ based on the *short vectors* algorithm has

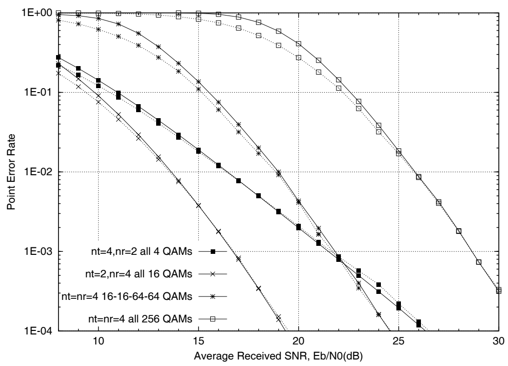

Fig. 2. Average error probability of a 4×2, 2×4 and 4×4 MIMO channels ($T_c = 10$). Analytic approximation (continuous lines) and Monte Carlo simulation (dotted lines)

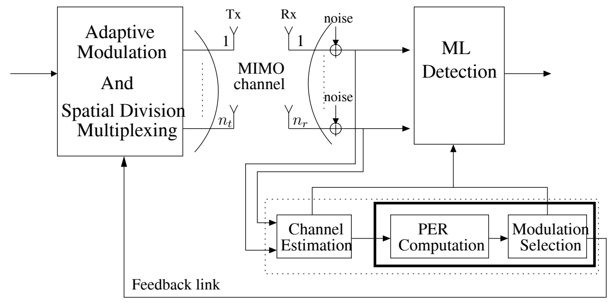

Fig. 3. Adaptive modulation scheme for MIMO channels

an exponential complexity in $n - m$, where $n = 2n_t$ and $m = 2n_r$. To reduce the search complexity for rank deficient MIMO system, we adapt to our problem the algorithm proposed in [4] for the generalization of the sphere decoder. The basic idea of this solution consists in forcing the positive-definiteness of the matrix $\mathbf{M}$. The search of the points $\mathbf{y} = \mathbf{zG}$ inside a sphere of radius C centered at the origin can be written as

$$\|\mathbf{y}\|^2 = \mathbf{zMz}^t = \mathbf{z}\tilde{\mathbf{M}}\mathbf{z}^t - \alpha\mathbf{zz}^t \leq C \tag{8}$$

where $\tilde{\mathbf{M}} = \mathbf{M} + \alpha\mathbf{I}_n$ is a definite-positive matrix. Inequality (8) is equivalent to

$$\mathbf{z}\tilde{\mathbf{M}}\mathbf{z}^t = \|\mathbf{z}\tilde{\mathbf{R}}\|^2 \leq C + \alpha\mathbf{zz}^t \leq C + \alpha \sum_{i=1}^{n} z^2_{\max_i} \tag{9}$$

where $\tilde{\mathbf{R}}$ is a $n \times n$ upper triangular matrix resulting of the cholesky factorization of $\tilde{\mathbf{M}}$ and $z_{\max_i} = \sqrt{M_{\lceil i/2 \rceil}} - 1$. Hence, it is possible to apply the *short vectors* algorithm to search the points $\tilde{\mathbf{x}} = \mathbf{z}\tilde{\mathbf{R}}$ belonging to the sphere centered at the origin of square radius $\tilde{C} = C + \alpha \sum_{i=1}^{n} z^2_{\max_i}$. This algorithm outputs all the integer components $\mathbf{z}$ verifying (9). Only those satisfying (8) are kept. Notice that the complexity of the proposed method depends on the value of the new radius $\tilde{C}$ that increases with the number of transmit antennas and especially with high constellation sizes. A judicious choice of the factor α reduces the complexity too. For simulation, α is fixed to 0.5.

4. APPLICATION TO ADAPTIVE MODULATION

Fig. 3 illustrates a MIMO system with n_t and n_r active transmit and receive antennas. At the receiver side, a channel estimation block provides $\mathbf{H}$ without error and N_0 to the adaptation block. The PER computation function employs (7) to compute $PER = Pe(C^{\mathbf{H}})$ for the different QAM combinations. For example, if N_q distinct QAMs are used, the PER should be computed N_q times if same QAM is applied on Tx antennas and $N_q^{n_t}$ times when distinct QAMs are used. The final block selects the optimal solution $(M_1, M_2, \ldots, M_{n_t})_{opt}$ that maximizes $\sum_{k=1}^{n_t} \log_2(M_k)$ under the constrain $PER \leq PER_{\text{target}}$. Finally, the transmitter is informed about the selection via a perfect feedback to adjust the modulations applied on the Tx antennas. The complexity of the adaptive scheme depends on the number of modulations to be tested in order to select the optimal one. The poor adaptive modulation when all QAMs are identical has a low adaptation complexity proportional to N_q. On the contrary, the efficient adaptive modulation when QAM constellations may be distinct per Tx antenna has an adaptation complexity proportional to $N_q^{n_t}$. The list of all QAM combinations can be reduced from $N_q^{n_t}$ down to $(N_q - 1)n_t + 1$ without performance loss. As example, for $n_t = N_q = 4$, we sort the transmit antennas such that $||H_1||^2 \leq \ldots \leq ||H_4||^2$, where H_i is the i^{th} row of $\mathbf{H}$ and $||H_i||^2 = \sum_{j=1}^{n_t} |h_{ij}|^2$. Then, we start from the most robust combination (all 4-QAMs) upward to the most efficient combination (all 256-QAMs). Only one integer is changed from one row to another according to a decreasing order of Tx antennas power. Thanks to the dichotomy method applied on the reduced list, a maximum of 4 evaluations of $Pe(C^{\mathbf{H}})$ are required instead of $N_q^{n_t} = 256$. Fig. 4 presents the performance of a 4×4 MIMO system in terms of error probability averaging over channel. The proposed adaptive modulation is shown in the middle surrounded by two extreme non-adaptive modulations, the all 4-QAMs and the all 256-QAMs, where $PER_{\text{target}} = 10^{-3}$. Clearly, the all 4-QAMs satisfies the constraints on PER but exhibits a severe loss in spectral efficiency. The all 256-QAMs has a maximum rate of 32 bits per channel use but does not guarantee the target PER in the whole range of interest $15dB \leq E_b/N_0 \leq 30dB$. The adaptive modulation policy leads to an optimization of the spectral efficiency while keeping the error probability close to the target.

5. APPLICATION TO ANTENNA SELECTION

The antenna selection scheme can be represented by a figure equivalent to Fig. 3. At the receiver side, the channel estima-

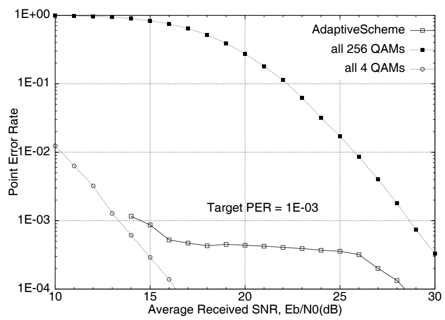

Fig. 4. Point Error Rate with the adaptive modulation policy, 4×4 MIMO channel ($T_c = 100$).

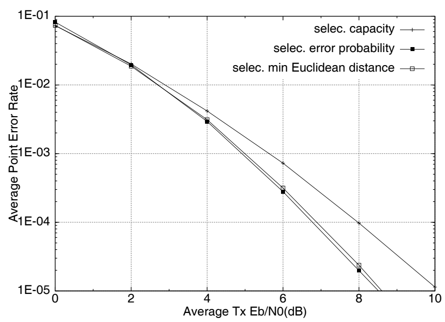

Fig. 5. Error probability with transmit antenna selection, 4×4 MIMO channel ($T_c = 100$). Select 4 transmit antennas among 5 Tx antennas with different selection criteria

tion and N_0 are provided to the PER computation block. The PER (7) is evaluated for $\left(\begin{array}{c} N_t \\ n_t \end{array}\right)$ $\left(\text{resp. } \left(\begin{array}{c} N_r \\ n_r \end{array}\right)\right)$ possible combinations of antennas in the case of transmit (resp. receive) selection, where N_t and N_r are the available Tx and Rx antennas. Then the final block selects the set of antennas at the transmitter and/or the receiver that minimizes the error probability. Finally, both blocks at the transmitter and the receiver, which are in charge of antennas activation at both sides, are perfectly informed about the selection.

The proposed selection scheme is compared to other existing selection criteria, namely capacity and minimum Euclidean distance of the receive constellation [5] in Fig. 5. We consider a 5×4 MIMO system with QPSK applied at the transmitter. The selection is applied only at the transmitter side. The selection consists in choosing the 4 best transmit antennas optimizing the selection criterion. In Fig. 5, the performance of 4×4 quasi-static MIMO channel, are depicted w.r.t the average transmit SNR, for different selection criteria. It is noticed that our proposed criterion minimizes the PER for all SNR range. For $PER = 10^{-4}$, the minimum Euclidean distance criterion presents a loss of about 0.5dB from our scheme, whereas a loss of 1.22dB is observed with the capacity criterion.

6. CONCLUSION

A new approximation for the conditional error probability on quasi-static multiple antenna channels has been described. For a fixed channel matrix $\mathbf{H}$, it is possible to predict the performance of QAM modulations transmitted over the MIMO channel in presence of an AWGN. The approximation is based on a tight union bound for the point error probability in the n-dimensional real space. As applications, we described an adaptive QAM modulation scheme for quasi-static MIMO channels and a new antenna selection criterion.

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