# ASYMPTOTIC ANALYSIS OF THE GAUSSIAN BROADCAST CHANNEL WITH PERTURBATION PREPROCESSING

*M. Stojnic, H. Vikalo, and B. Hassibi*

California Institute of Technology, Pasadena, CA
e-mail: mihailo,hvikalo,hassibi@systems.caltech.edu

## ABSTRACT

The sum rate capacity of the multi-antenna Gaussian broadcast channel has recently been computed. However, the search for computationally efficient practical schemes that achieve it is still in progress. When the channel state information is fully available at the transmitter, the dirty paper coding (DPC) technique is known to achieve the maximal throughput, but is computationally infeasible. In this paper, we analyze the asymptotic behavior of one of its alternatives – the recently suggested so-called vector perturbation technique. We show that for a square channel, where the number of users is large and equal to the number of transmit antennas, its sum rate approaches that of the DPC technique. More precisely, we show that at both low and high signal-to-noise ratio (SNR), the scheme under consideration is asymptotically optimal. Furthermore, we obtain similar results in the case where the number of users is much larger than the number of transmit antennas.

## 1. INTRODUCTION

The limits of performance of multi-antenna Gaussian broadcast channel have recently been extensively studied (see, e.g., [1], [2], and the references therein). It has been shown that when the channel state information (CSI) is fully available at the transmitter, the so-called dirty paper coding (DPC) technique achieves the capacity of multi-antenna broadcast channel [3]. However, the DPC scheme is exponentially complex and appears to be difficult to implement in practical systems. To this end, various heuristics with suboptimal performance but efficient implementation have recently been proposed. In [4], vector quantization is used in combination with powerful coding schemes to achieve a large fraction of the promised capacity. In [5], a technique referred to as the vector perturbation technique (VPT) was proposed, and further considered in [6]. Simulation results presented there indicate that the proposed technique achieves performance close to the optimal one.

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In this paper, we analyze the theoretical limits of the VPT [5]. In particular, we show that when the number of users in the broadcast system is large, the sum rate achievable by the VPT approaches the sum rate achievable by the DPC scheme, both in the low and the high SNR regime. While the scheme introduced in [5] and further studied in [6] is practically feasible, the worst case complexity of its implementation is still exponential. On the other hand, our proof for lower-bounding the asymptotical sum-rate performance of the VPT is constructive and based on an algorithm that is polynomial in the number of users.

We assume the standard broadcast channel model,

$$\mathbf{y} = H\mathbf{s} + \mathbf{v}, \tag{1}$$

where $H$ is a $K \times M$ matrix whose entries are independent, identically distributed (i.i.d.) complex Gaussian random variables $\mathcal{C}_{\mathcal{N}}(0,1)$, $K$ is the number of users, $M$ is the number of transmit antennas, $\mathbf{v}$ is a $K \times 1$ noise vector whose entries are independent of entries in $H$ and i.i.d. Gaussian random variables with zero-mean and $\sigma^2 = 1/\rho$ variance, and $\mathbf{s}$ is an $M \times 1$ vector which is transmitted over the channel. Furthermore, we impose the constraint $E\|\mathbf{s}\|^2 = 1$; hence, the receivers do not need to know instantiations of the channel. (The case $\|\mathbf{s}\|^2 = 1$, considered in [5], can be treated similarly and leads to similar results.)

Since we focus on analyzing the asymptotic performance of the vector perturbation technique, we start by reviewing it in the next section.

## 2. THE VECTOR-PERTURBATION TECHNIQUE

Following [5], we consider the scenario where the number of antennas on transmitter is equal to the number of users, i.e., $K = M$. [Later in the paper we will consider the vector perturbation technique for $K \gg M$ and generalize our results to that case.] Furthermore, we assume that the entries of the $K \times 1$ symbols vector $\mathbf{u}$ intended for the users are the points in a QAM constellation.

The vector perturbation technique [5] relates the transmitted vector $\mathbf{s}$ to the information vector $\mathbf{u}$ as follows,

$$\mathbf{s} = \frac{H^{-1}(\mathbf{u} + \tau\hat{\mathbf{l}})}{\sqrt{E\|H^{-1}(\mathbf{u} + \tau\hat{\mathbf{l}})\|^2}}, \tag{2}$$

where $\tau$ is an *a priori* determined positive constant, and where $\hat{\mathbf{l}}$ is the solution to the following optimization problem

$$\hat{\mathbf{l}} = \text{argmin}_{\mathcal{R}(\mathbf{l})\in Z^M, \mathcal{I}(\mathbf{l})\in Z^M} ||H^{-1}(\mathbf{u}+\tau\mathbf{l})||^2. \quad (3)$$

Note that $\mathcal{R}(\mathbf{l})$ and $\mathcal{I}(\mathbf{l})$ denote the real and the imaginary part of the vector $\mathbf{l}$, respectively. The main idea behind (3) is to eliminate (or to minimize) the power penalty which happens in the case if the so-called zero-forcing (ZF) scheme (obtained for $\hat{\mathbf{l}} = 0$ in (2)) is applied.

The signals received by the $k^{th}$ user are of the form

$$\mathbf{y}_i = \frac{u_i + \tau\hat{l}_i}{\sqrt{E||L^{-1}(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}} + v_i, \quad 1 \le i \le K, \quad (4)$$

where $L$ is a lower-triangular matrix in the $LQ$-decomposition of $H$, i.e., $H = LQ$, and $Q$ is a unitary matrix. Decoding of these signals is simple, and the only processing required from the receivers is scaling by $\sqrt{E||H^{-1}(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}$ [5].

## 3. CASE $K = M$

In this section, we analyze the VPT for $K = M$. Before proceeding any further, we slightly modify the perturbation technique as follows. Let $D$ be a diagonal matrix such that

$$D = \text{diag}(L_{1,1}^{1+\beta}, L_{2,2}^{1+\beta}, \ldots, L_{n,n}^{1+\beta}), \quad (5)$$

where $\beta$ is any integer such that $\beta \ge 0$. Instead of transmitting $\mathbf{s}$ as given by (2), we define $\mathbf{s}$ to be

$$\mathbf{s} = \frac{H^{-1}D(\mathbf{u}+\tau\hat{\mathbf{l}})}{\sqrt{E||H^{-1}D(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}}. \quad (6)$$

Consequently, the signal received at the $i^{th}$ user becomes

$$\mathbf{y}_i = L_{i,i}^{1+\beta}\frac{u_i + \tau\hat{l}_i}{\sqrt{E||L^{-1}D(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}} + v_i, \; 1 \le i \le K. \quad (7)$$

We refer to this scheme as the diagonal vector perturbation technique (DVPT). From (7) it follows that the sum-rate of the DVPT can then be computed as a summation of the sum-rates of K decoupled channels,

$$R_{DVPT} = E\sum_{i=1}^{K}\log\Big(1 + L_{i,i}^{2(1+\beta)}\frac{\rho||u_i+\tau\hat{l}_i||^2}{E||L^{-1}D(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}\Big). \quad (8)$$

We are interested in bounding the value of $R_{DVPT}$; to this end, it will be useful to first derive a few inequalities.

Although the use of the VPT on broadcast channels performs well in practice, it requires solving (3), which is NP-hard. Use of the sphere decoding (or any other) algorithm may often be infeasible. Therefore, we employ a heuristic nulling and canceling [8] technique to solve (3). To this end, let us denote $B = L^{-1}D$. Clearly, $B_{i,i} = L_{i,i}^{\beta}$. Generate $\mathbf{l}^{lb} = \mathbf{l}^{lbr} + j\mathbf{l}^{lbc}$ according to the nulling and canceling procedure as follows,

$$\begin{aligned}
\mathbf{l}_1^{lbr} &= \lfloor -\frac{\mathcal{R}(u_1)}{\tau}\rfloor \\
\mathbf{l}_1^{lbc} &= \lfloor -\frac{\mathcal{I}(u_1)}{\tau}\rfloor \\
\mathbf{l}_2^{lbr} &= \lfloor -\frac{B_{2,2}\mathcal{R}(u_2) + \mathcal{R}(B_{2,1}(u_1+\tau l_1^{lb}))}{B_{2,2}\tau}\rfloor \\
\mathbf{l}_2^{lbc} &= \lfloor -\frac{B_{2,2}\mathcal{I}(u_2) + \mathcal{I}(B_{2,1}(u_1+\tau l_1^{lb}))}{B_{2,2}\tau}\rfloor \\
&\vdots \\
\mathbf{l}_K^{lbr} &= \lfloor -\frac{B_{K,K}\mathcal{R}(u_K) + \sum_{i=1}^{K-1}\mathcal{R}(B_{K,i}(u_i+\tau l_i^{lb}))}{B_{K,K}\tau}\rfloor \\
\mathbf{l}_K^{lbc} &= \lfloor -\frac{B_{K,K}\mathcal{I}(u_K) + \sum_{i=1}^{K-1}\mathcal{I}(B_{K,i}(u_i+\tau l_i^{lb}))}{B_{K,K}\tau}\rfloor.
\end{aligned}$$

Since for any two real numbers $a$ and $b$ holds that $|a - b\lfloor\frac{a}{b}\rfloor|^2 \le b^2$, we obtain

$$|u_i + \tau l_i^{lb}|^2 \le 2\tau^2 B_{i,i}^2. \quad (9)$$

Careful examination of the previous procedure reveals that any time we obtain $l_i^{lbr} = 0$, we can change it to either $l_i^{lbr} = 1$ or $l_i^{lbr} = -1$ and still preserve the validity of (9). Thus in addition to (9), we can also establish a lower bound on $|u_i + \tau l_i^{lb}|^2$ depending on the sign of $u_i$ or $l_i^{lb}$,

$$\begin{aligned}
&|u_i + \tau l_i^{lb}|^2 \ge |-|\mathcal{R}(u_i)| - j|\mathcal{I}(u_i)| + \tau(1+j)|^2 \ge \\
&|-\text{max}_i|\mathcal{R}(u_i)| - j\text{max}_i|\mathcal{I}(u_i)| + \tau(1+j)|^2 = \zeta. \quad (10)
\end{aligned}$$

We may now begin our derivation of a bound on $R_{DVPT}$. To facilitate fluent presentation, we first treat the low SNR case, and then generalize the results to any SNR.

### 3.1. Low SNR regime ($\rho \to 0$)

For $\rho \to 0$, we have

$$\begin{aligned}
R_{DVPT} &= E\sum_{i=1}^{K}\log\Big(1 + L_{i,i}^{2(1+\beta)}\frac{\rho||u_i+\tau\hat{l}_i||^2}{E||L^{-1}D(\mathbf{u}+\tau\hat{\mathbf{l}})||^2}\Big) \\
&= K + \rho\frac{\sum_{i=1}^{K} EL_{i,i}^{2(1+\beta)}||u_i+\tau\hat{l}_i||^2}{E||B(\mathbf{u}+\tau\hat{\mathbf{l}})||^2} + \mathcal{O}(\rho^2).
\end{aligned}$$

Using (10) to lower bound the numerator and (9) to upper bound the denominator of the fraction in the expression above, and using the fact that $L_{i,i}^2$ are i.i.d. random variables with $\chi^2_{2(K-i+1)}$ distribution, we have

$$R_{DVPT} \geq K + \rho \frac{\zeta \sum_{i=1}^{K} E(L_{i,i}^2)^{1+\beta}}{2\tau^2 \sum_{i=1}^{K} E(L_{i,i}^2)^{\beta}} + \mathcal{O}(\rho^2).$$

$$= K + \rho \frac{\zeta \sum_{i=1}^{K} \prod_{k=1}^{1+\beta}(2(K+1-i)+2(k-1))}{2\tau^2 \sum_{i=1}^{K} \prod_{k=1}^{\beta}(2(K+1-i)+2(k-1))} + \mathcal{O}(\rho^2)$$

$$\geq K + \rho \frac{\zeta \sum_{i=1}^{K}(2(K+1-i))^{1+\beta}}{2\tau^2 \sum_{i=1}^{K}(2(K+1-i)+2(\beta-1))^{\beta}} + \mathcal{O}(\rho^2) \quad (11)$$

Therefore, we can write

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K} \geq 1 + 2\rho \frac{\zeta(1+\beta)}{2\tau^2(2+\beta)}. \quad (12)$$

Let $w$ be the width of a QAM constellation,i.e. let $w = 2\max_{\mathbf{u}} \max\{\max_i |\mathcal{R}(u_i)|, \max_i |\mathcal{I}(u_i)|\}$. Clearly, $\zeta \geq 2(\tau - \frac{w}{2})^2$. Then for $\tau \gg \frac{w}{2}$, and for $\beta \to \infty$, $K \gg \beta$, we can write

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K} \geq 1 + 2\rho. \quad (13)$$

The results stated in (11), (12), and (13), imply that the sum-rate of the diagonal vector perturbation technique scales linearly with the number of users. In fact, this result may be established directly from (11). However, in order to tighten the coefficients in front of $\rho$, we derived (13) as well.

We summarize our results in the following theorem.

**Theorem 1** *Consider communication in low SNR regime ($\rho \to 0$) over a square Gaussian broadcast channel using the diagonal vector perturbation technique with parameters $\beta \geq 1$ and $\tau \geq w$, where $w$ is the width of a QAM constellation. Then*

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K} \geq 1 + 2\rho(1 - \frac{w}{2\tau})^2 \frac{1+\beta}{2+\beta}.$$

**Proof:** Follows from the discussion above. ■

**Corollary 1** *Let all assumptions of Theorem 1 hold. Furthermore, let $\tau \gg w$, $\beta \to \infty$, and $\frac{K}{\beta} \gg 1$. Then*

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K} \geq 1 + 2\rho.$$

### 3.2. General SNR

For simplicity, in this subsection we fix $\beta = 0$. Similar to the procedure in Section 3.1, we use (10) to lower bound the numerator and (9) to upper bound the denominator of the fraction in the expression given in (8),

$$R_{DVPT} = E \sum_{i=1}^{K} \log\Big(1 + L_{i,i}^2 \frac{\rho ||u_i + \tau \hat{l}_i||^2}{E||L^{-1}D(\mathbf{u} + \tau\hat{\mathbf{l}})||^2}\Big)$$

$$\geq E \sum_{i=1}^{K-1} \log(1 + \rho \frac{\zeta \hat{L}_{i,i}^2}{2\tau^2 K}) = E\log \prod_{i=1}^{K-1}(1 + \rho \frac{\zeta}{2\tau^2 K} L_{i,i}^2)$$

Applying the arithmetic-geometric mean inequality, it can easily be shown that

$$\prod_{i=1}^{K-1}(1 + \rho \frac{\zeta}{2\tau^2 K} L_{i,i}^2) \geq (1 + \rho \frac{\zeta}{2\tau^2 K}(\prod_{i=1}^{K-1} L_{i,i}^2)^{\frac{1}{K-1}})^{K-1}$$

Then, we have

$$R_{DVPT} \geq (K-1)E\log(I + \rho \frac{\zeta}{2\tau^2 K}(\prod_{i=1}^{K-1} L_{i,i}^2)^{\frac{1}{K-1}})$$

$$\geq (K-1)\log\Big(1 + \rho \frac{\zeta \prod_{i=1}^{K-1}(E((L_{i,i}^2)^{-1}))^{-\frac{1}{K-1}}}{2\tau^2 K}\Big). \quad (14)$$

Using the fact that $L_{i,i}^2$ are i.i.d. random variables with $\chi^2_{2(K-i+1)}$ distributions, and the Stirling's formula to approximate the factorial, we obtain

$$R_{DVPT} \geq (K-1)\log\Big(1 + \frac{\rho\zeta(\prod_{i=2}^{K} 2(K-i+1))^{\frac{1}{K-1}}}{2\tau^2 K}\Big)$$

$$\geq (K-1)\log(1 + \frac{2\rho\zeta(K-1)!^{\frac{1}{K-1}}}{2\tau^2 K}) \geq (K-1)\log\Big(1 + \frac{2\rho\zeta}{2\tau^2 e}\Big)$$

$$\geq (K-1)\log\Big(1 + \frac{2\rho}{e}(1 - \frac{w}{2\tau})^2\Big) \quad (15)$$

It is worth pointing out that for $\rho \to \infty$ we can also upper bound the value of $R_{DVPT}$. Instead of (14), using Jensen's and the arithmetic-geometric mean inequalities we can write

$$R_{DVPT} \leq E\log\Big(\rho^K \frac{\prod_{i=1}^{K} L_{i,i}^2 \prod_{i=1}^{K} ||u_i + \tau\hat{l}_i||^2}{(\sum_{i=1}^{K} ||u_i + \tau\hat{l}_i||^2)^K}\Big)$$

$$\leq \log\Big(\rho^K \frac{\prod_{i=1}^{K} EL_{i,i}^2}{K^K}\Big) \leq \log\Big((2\rho)^K \frac{K!}{K^K}\Big)$$

$$\leq K\log(\frac{2\rho}{e}) + \mathcal{O}(\log K). \quad (16)$$

The results from this subsection are summarized in the following theorem.

**Theorem 2** *Consider communication over square Gaussian broadcast channel using the diagonal vector perturbation technique with parameters $\beta = 0$ and $\tau > \frac{w}{2}$, where $w$ is the width of a QAM constellation. Then*

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K log\Big(1 + \frac{2\rho}{e}(1 - \frac{w}{2\tau})^2\Big)} \geq 1.$$

**Proof:** Follows from (15). ■

**Corollary 2** *Let assumptions of Theorem 2 hold. Also let $\rho \to \infty$. Then*

$$\lim_{K\to\infty} \frac{R_{DVPT}}{K log\rho} = 1.$$

Theorems 1 and 2 imply that when the diagonal vector perturbation technique (with appropriate parameters) is employed for communication over Gaussian broadcast channel, the sum-rate scales linearly with the number of users. Furthermore, in high SNR regime the scaling law is not only liner in the number of users, but also optimal, i.e. equal to that of the capacity achieving DPC technique.

## 4. CASE $K \gg M$

In this section, we study the asymptotic behavior of the VPT and DVPT schemes for $K \gg M$. We should point out that this regime (in particular, the case $\frac{\log K}{M} \geq const., K \to \infty$) was considered in [9], where it was shown that, in limit, the maximum throughput may be achieved with only partial CSI at the transmitter. The VPT and DVPT, on the other hand, require full CSI; however, since we have shown that these simple schemes asymptotically achieve the maximum throughput when the number of transmit antennas and users is the same, it is of interest to extend these results to $K \gg M$ case as well.

A generalization of the results to $K \gg M$ case is relatively straightforward. In particular, at any transmission interval we select a subset of $M$ users to which we transmit. Define $H_{(k)} = H_{(k-1)M+1:kM,(k-1)M+1:kM}$. Let $\lambda_k^{min}$ be the minimal eigenvalue of the matrix $H_{(k)}^* H_{(k)}$, and let $\xi = \arg\max_{k\in\{1,2,\ldots,\lfloor\frac{K}{M}\rfloor\}} \lambda_k^{min}$. Then we transmit to the users $(\xi-1)M+1, (\xi-1)M+2, \ldots, \xi M$, employing the DVPT with $H_{(\xi)}$. Let $\hat{L}$ be a lower-triangular matrix from the LQ-decomposition $H_{(\xi)} = \hat{L}Q$, where $Q$ is unitary. Then, instead of (14) we can write

$$R_{DVPT} = E\sum_{i=1}^{M} \log(1+\rho\frac{\zeta \hat{L}_{i,i}^2}{2\tau^2 M}) \geq M\log\Big(1+\rho\frac{\zeta}{2\tau^2 M E((\lambda_\xi^{min})^{-1})}\Big).$$

Further, using results from extreme value theory, it can be shown that $\lim_{\frac{K}{M}\to\infty} E((\lambda_\xi^{min})^{-1}) \to \frac{M}{2\log\frac{K}{M}}$ (see, e.g. [9]). The results from this section are summarized in the following theorem.

**Theorem 3** *Consider communication over tall Gaussian broadcast channel ($\frac{K}{M}\to\infty$) using the diagonal vector perturbation technique with parameters $\beta = 0$ and $\tau > \frac{w}{2}$, where $w$ is the width of a QAM constellation. Then*

$$\lim_{\frac{K}{M}\to\infty} \frac{R_{DVPT}}{M log\Big(1+\frac{2\rho}{M^2} log\frac{K}{M}\Big)} \geq 1.$$

**Proof:** Follows from the discussion above. ■

In case of high SNR ($\rho \to \infty$) we have the following corollary.

**Corollary 3** *Let assumptions of Theorem 3 hold. Also let $\rho\to\infty$. Then*

$$\lim_{\frac{K}{M},\rho\to\infty} \frac{R_{DVPT}}{M log(\rho log K)} \geq 1.$$

Previous corollary says that the sum-rate of the VPT asymptotically achieves the same sum-rate as the DPC.

**Remark:** We point out that using the same selection of users as suggested in this section, it is easy to show that, under the assumptions of the previous corollary, even the ZF scheme asymptotically achieves the same sum-rate as the DPC.

## 5. CONCLUSION

In this work, we studied the asymptotic performance of the achievable throughput on the Gaussian broadcast channel with vector perturbation preprocessing. We derived explicitly the achievable sum-rate scaling laws in the case when the perturbation preprocessing is applied at the transmitter. As it turns out, those scaling laws are matching the already known capacity achieving scheme (DPC) scaling laws in the case when the CSI is available at the transmitter. Furthermore, unlike the DPC, our scheme is simple and can be implemented in polynomial time.

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