SUBF vs. MUBF in a Gaussian MIMO Broadcast Channel with Partial Channel State Information

Youngjae Kim and John M. Cioffi
Department of Electrical Engineering
Stanford University
Stanford, CA 94305
Email: {youngjae, cioffi}@stanford.edu

Ravi Narasimhan
Department of Electrical Engineering
University of California.
Santa Cruz, CA 95064
Email: ravi@soe.ucsc.edu

***Abstract*— In a Gaussian MIMO broadcast channel, serving more than one user at a time is generally more beneficial in terms of average sum-rate. Under partial Channel State Information (CSI) feedback constraints, however, serving only one user at a time can be a better multiple access method when the number of active users in the system is smaller than a certain threshold. This paper uses Single User Beam-forming (SUBF) to refer to the transmitting scheme that serves only one user at a time and Multi-user Beam-forming (MUBF) to refer to the scheme without such a constraint, i.e., a scheme that is allowed to serve more than one user at a time. It turns out that the aforementioned user threshold varies with other system parameters, especially, it increases as the number of transmit antennas increases. Monte Carlo simulations and a statistical analysis are provided to verify the results.**

I. Introduction

The use of multiple antennas in transmitters and/or receivers has been studied considerably. In a general Multi-Input Multi-Output (MIMO) Broadcast Channel (BC), it has been discovered that the sum capacity can be achieved by Dirty Paper Coding (DPC) [1], [2]. Although it is optimal in terms of sum-rate, it is hard to implement and requires perfect CSI at the transmitter side, which is practically very challenging.

This paper considers MIMO BC with partial CSI feedback constraints. Partial CSI means any imperfect *and* un-quantized feedback information about the channel such as Signal-to-Interference-and-Noise Ratio (SINR), a power distribution profile, achievable rates and so on. In terms of feedback amount, perfect CSI requires $2(M_T \cdot M_R)$ real numbers per channel realization since the channel H is an $M_R \times M_T$ complex matrix, where M_T is the number of transmit antennas and M_R is the number of receive antennas. Thus, partial CSI feedback information consists of 1 to $2(M_T \cdot M_R) - 1$ real numbers.

This paper considers three such schemes in MIMO BC. In two schemes, the transmitter serves only one user at a time, which is referred to as SUBF [3] in this paper. These two SUBF schemes are different from each other in terms of feedback information amount. In the third one, the transmitter has no such constraint and is allowed to serve more than one user at a time, which is referred to as MUBF.

In the first SUBF, each receiver estimates its own achievable rate and feeds it back to the transmitter. Upon receiving all the achievable rates from the receivers, the transmitter selects the best receiver and sends data only to that receiver. By doing this, the transmitter can serve only the best receiver at any time and ride the peak of the fading channel, which is so-called multi-user diversity [4]. In addition, the receiver may calculate the optimal power distribution profile and feeds it back to the transmitter along with the achievable rate. The second SUBF scheme does this and is very beneficial in low SNR regions. In MUBF, the transmitter sends M_T orthogonal data streams (or beams) to all the receivers. Each receiver estimates the SINR of each stream and sends them back to the transmitter with the stream indices. Then, the transmitter selects the best receiver in each stream independently and sends data only to the selected receivers through the corresponding streams. Different receivers may be selected in different streams, thus serving more than one user at a time is possible.

Unless the broadcast channel is degraded, which is true when $M_T = 1$ [5], the optimal DPC scheme usually serves more than one user at a time. Thus, it can be easily imagined that MUBF is better than SUBF when the number of users in the system is large. When there are a small number of users in the system, however, this paper demonstrates SUBF is more beneficial than MUBF in terms of sum-rate. This is true until the number of users reaches some threshold. This threshold is related with other system parameters such as M_T, M_R or Signal-to-Noise-Ratio (SNR). In particular, it increases as M_T increases. As will be detailed in the paper, the main reason for this phenomenon is because MUBF should consider other streams as noise because they are intended for other users. In SUBF, however, all the streams are intended for one user, thus each stream can be more efficiently detected by joint signal processing.

This paper is organized as follows. Section II introduces the system model considered in the paper, Section III and IV review three schemes in MIMO BC with partial CSI feedback constraints, derives the random variable for the sum-rate in each scheme and compares them and Section V provides the simulation results. Section VI finds the threshold using an approximated analysis and Section VII concludes the paper.

II. SYSTEM MODEL

The Gaussian broadcast system considered in this paper consists of one Base Station (BS) and U users. The signal received by the u^{th} user is $\underline{y}_u (\in \mathbb{C}^{M_R \times 1})$ and is represented as follows:

$$\underline{y}_u = \sqrt{\frac{\rho}{M_T}}\mathbf{H}_u \underline{x} + \underline{n}_u, \quad (1)$$

where $\mathbf{H}_u (\in \mathbb{C}^{M_R \times M_T})$ is the channel from the BS to the u^{th} user, $\underline{x} (\in \mathbb{C}^{M_T \times 1})$ is the transmitting signal and $\underline{n}_u (\in \mathbb{C}^{M_R \times 1})$ is the additive white Gaussian noise. For a rich scattering environment, each element of $\mathbf{H}_u$ is modeled as an i.i.d. complex Gaussian random variable with zero mean and unit variance. The channel is assumed to be constant within one packet time and changes independently in the next packet time, i.e., *block fading model*. Without loss of generality, $\mathbb{E}[\underline{n}_u \underline{n}_u^*] = \mathbf{I}$ such that ρ is SNR. The BS has a total transmitting power constraint, $\text{Tr}(\mathbf{R}_{\underline{xx}}) \le M_T$, where $\mathbf{R}_{\underline{xx}} = \mathbb{E}[\underline{x}\underline{x}^*]$ and $[\cdot]^*$ means a conjugate-transpose operation. Throughout the paper, underlined letters represent vectors, bold-face letters represent matrices and plain letters represent scalars.

III. SUBF SCHEMES

Denote the maximum achievable rate of the u^{th} user under any given constraints as $R_{u,(\cdot)}$. $R_{u,MAX-SUBF}$, for instance, represents the achievable rate of the u^{th} user in SUBF, i.e., one user at a time, when perfect transmitter CSI is available. Thus, the problem of finding $R_{u,MAX-SUBF}$ can be cast into the following optimization form:

$$R_{u,MAX-SUBF} = \max_{\mathbf{R}_{\underline{xx}}} \log_2 |\mathbf{I_{M_R}} + \frac{\rho}{M_T}\mathbf{H}_u \mathbf{R}_{\underline{xx}} \mathbf{H}_u^*| \quad (2)$$

$$\text{subject to } \text{Tr}(\mathbf{R}_{\underline{xx}}) \le M_T, \mathbf{R}_{\underline{xx}} \succeq 0, \quad (3)$$

where $\mathbf{X} \succeq 0$ means that $\mathbf{X}$ is a positive semi-definite matrix. Since the BS would select the best user in order to maximize the sum rate at any time, the selected rate for this scheme would be

$$R_{MAX-SUBF} = \max_{u=1,\cdots,U} R_{u,MAX-SUBF}, \quad (4)$$

which could be considered as the upper bound of any SUBF schemes that use partial CSI feedback.

Secondly, if any signal coordination among different transmit antennas is not allowed but the power per each antenna can be optimized, the constraint of $R_{\underline{xx}}$ being diagonal is added to the rate optimization problem. That is,

$$R_{u,SUBF} = \max_{\mathbf{R}_{\underline{xx}}} \log_2 |\mathbf{I_{M_R}} + \frac{\rho}{M_T}\mathbf{H}_u \mathbf{R}_{\underline{xx}} \mathbf{H}_u^*| \quad (5)$$

$$\text{subject to } \text{Tr}(\mathbf{R}_{\underline{xx}}) \le M_T, \mathbf{R}_{\underline{xx}} \succeq 0, \quad (6)$$

$$\mathbf{R}_{\underline{xx}} \text{ is } \textit{diagonal} \quad (7)$$

In [3], it was observed that this problem is equivalent to the Multiple-Access-Channel (MAC) sum capacity problem with a sum power constraint M_T, which can be efficiently solved by the dual decomposition approach and Iterative Water-filling by Yu [6]. After the selection process by the BS, the sum rate in this case will be

$$R_{SUBF} = \max_{u=1,\cdots,U} R_{u,SUBF}. \quad (8)$$

Finally, if only the achievable rate is fed back, a BS has no choice but using $\mathbf{R}_{\underline{xx}} = \mathbf{I}$, then

$$R_{SUBF-RateOnly} = \max_{u=1,\cdots,U} R_{u,SUBF-RateOnly} \quad (9)$$

$$= \max_{u=1,\cdots,U} \log_2 |\mathbf{I_{M_R}} + \frac{\rho}{M_T}\mathbf{H}_u \mathbf{H}_u^*| \quad (10)$$

In [7], it was discovered that $R_{u,SUBF-RateOnly}$ can be well approximated by a Gaussian random variable. In particular, as $M_R \to \infty$, it was shown that

$$R_{u,SUBF-RateOnly} \sim \mathcal{N}\left(M_T \log_2(1 + \frac{M_R \cdot \rho}{M_T}), \frac{M_T \log_2^2 e}{M_R}\right) \quad (11)$$

Although this expression is true only in an asymptotic region, Gaussian approximation itself still holds very good even with small M_R, which implies that the knowledge of the mean and variance is sufficient to determine the distribution of $R_{u,SUBF-RateOnly}$. Since $R_{SUBF-RateOnly}$ is the maximum of i.i.d Gaussian random variables, its Cumulative Distribution Function (CDF) can be approximated by $P(R_{SUBF-RateOnly} \le z) = P(R_{u,SUBF-RateOnly} \le z)^U$.

Table I summarizes the schemes that are introduced in this paper based on the amount of feedback information. It is clear that $R_{MAX-SUBF} \ge R_{SUBF} \ge R_{SUBF-RateOnly}$ because of the weaker constraints for $R_{MAX-SUBF}$ compared to that of R_{SUBF} and so on. In the high SNR regions, however, it is well known that the power optimization is not beneficial [8], which leads that $R_{MAX-TDMA} \approx R_{SUBF} \approx R_{SUBF-RateOnly}$. Thus, when ρ is high, it is sufficient to compare only $R_{SUBF-RateOnly}$ with the MUBF scheme that will be explained in the next section.

IV. MUBF SCHEME WITH PARTIAL CSI FEEDBACK

After a transmitter sends M_T orthogonal beams using $R_{\underline{xx}} = \mathbf{I}$ and the receiver performs Minimum-Mean-Squared-Error Linear-Equalization (MMSE-LE), the unbiased SINR on the i^{th} stream of the u^{th} user can be expressed as [9]:

$$SINR_{i,u} = \frac{1}{\left[\left(\mathbf{I}_{M_T} + \frac{\rho}{M_T}\mathbf{H}_u^*\mathbf{H}_u\right)^{-1}\right]_{i,i}} - 1, \quad (12)$$

where $[\cdot]_{i,i}$ means the i^{th} diagonal element of the matrix.

BS selects the best user in each stream and the sum of those selected rates is the variable of interest, which can be

TABLE I
COMPARISON OF THE SCHEMES

CSI	Perfect	Partial (Rate+Power Profile)	Partial (each stream's SINR)	Partial (Rate Only)	None (Round Robin)
Rate	$R_{MAX-SUBF}, R_{DPC}$	R_{SUBF}	R_{MUBF}	$R_{SUBF-RateOnly}$	$R_{Round-Robin}$
Constraint at Tx.	Tr$(R_{\underline{xx}}) \leq M_T$	Tr$(R_{\underline{xx}}) \leq M_T$, $R_{\underline{xx}}$ diagonal	$R_{\underline{xx}} = \mathbf{I}$	$R_{\underline{xx}} = \mathbf{I}$	$R_{\underline{xx}} = \mathbf{I}$
CSI amount (Number of real numbers)	$2(M_T \cdot M_R)$	M_T+1	M_T	1	0
Multi-user Diversity	Yes	Yes	Yes	Yes	No

represented as:

$$R_{MUBF} = \sum_{i=1}^{M_T} R_{MUBF,i} = \sum_{i=1}^{M_T} \left(\max_{u=1,\cdots,U} \log_2(1 + SINR_{i,u}) \right) \tag{13}$$

$$= \sum_{i=1}^{M_T} \left(\log_2(1 + \max_{u=1,\cdots,U} SINR_{i,u}) \right) \tag{14}$$

(14) holds because $\log_2(1+x)$ is a monotonically increasing function. This scheme is called Multi-User Beam-forming (MUBF) compared to SUBF in this paper.

Interestingly, even without receiver coordination, the asymptotic growth rate of $\mathbb{E}[R_{MUBF}]$ as U increases turns out to be equal to that of $\mathbb{E}[R_{DPC}]$ [10], where R_{DPC} denotes the achievable rate by DPC and $\mathbb{E}[\cdot]$ here means averaging the variable over all the channel realizations, i.e., *an ergodic mean*. This fact strongly manifests the importance of partial CSI feedback. However, note that this is an asymptotic result and there always exists a gap between $\mathbb{E}[R_{DPC}]$ and $\mathbb{E}[R_{MUBF}]$. This is why SUBF can be better than MUBF in the system with a small U. Furthermore, from Table I, note that $R_{SUBF-RateOnly}$ uses even less amount of feedback information than R_{MUBF} but can outperform R_{MUBF} in some situations.

In a high SNR regime, Zero-Forcing (ZF)-LE performs almost as well as MMSE-LE, so $SINR_{i,u}$ can be well approximated by $SINR_{i,u} \approx SINR_{ZF,i,u} = \frac{1}{\left[\left(\frac{\rho}{M_T}\mathbf{H}_u^*\mathbf{H}_u\right)^{-1}\right]_{i,i}}$. It is known that $SINR_{ZF,i,u}$ is a chi-squared random variable with $2(M_R + M_T - 1)$ degrees of freedom [9]. Thus, when ρ is high, the CDF of $R_{MUBF,i}$ can be found numerically by using the cdf of a chi-squared distribution.

V. Numerical Results

Fig. 1 illustrates the ergodic sum rates along with U as both M_T and M_R increase from 2 to 4 when ρ = 20(dB). Here, U means the number of users who simultaneously request data transmission from BS at a specific time. In all the simulations, a capacity achieving Gaussian code is assumed to be used. First, note that since this is in a high SNR regime, the plot of $\mathbb{E}[R_{MAX-SUBF}]$ is almost identical to $\mathbb{E}[R_{SUBF-RateOnly}]$. Second, the multi-user diversity effect can be observed, i.e., the rates increase as U increases. Since SUBF is optimized to

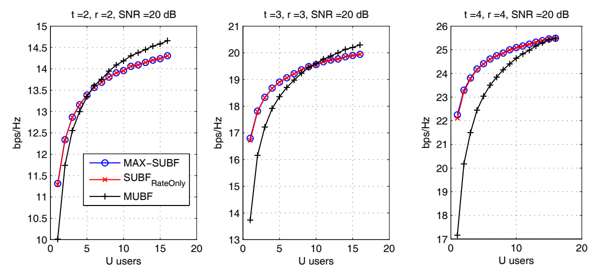

Fig. 1. Ergodic sum rates with partial CSI (ρ = 20dB).

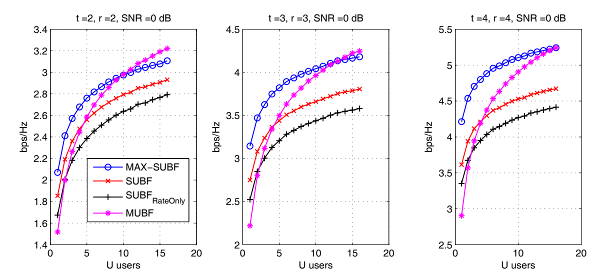

Fig. 2. Ergodic sum rates with partial CSI (ρ = 0dB).

serve only one user under given constraints, SUBF is clearly better than MUBF when $U = 1$. In more detail, all the streams in SUBF are intended for one user. Thus, the receiving user can use a Successive-Interference-Cancelation (SIC) technique that is exactly what the common receiver in a MAC would do for multi-user detection. On the other hand, MUBF just uses MMLE-LE decoding. This is the main reason that SUBF is better than MUBF when $U = 1$. In addition, the growth rate of $\mathbb{E}[R_{MUBF}]$ is higher than that of $\mathbb{E}[R_{SUBF-RateOnly}]$, thus there always exists a *cross-over point* where two curves meet. This cross-over point is a threshold under which SUBF is better than MUBF. Third, note that this threshold increases as M_T and M_R increase. This is because the performance of MUBF is limited by interferences among different data streams but SUBF can cancel out these interferences by joint signal processing at the receiver as explained above. From the plot, it can be observed that these thresholds are U = 6, 9, 16, when $M_T = M_R$ =2,3,4, respectively. In an actual BS, there are 30 users associated with it. Therefore, it is possible that fewer than 10 users request data transmission at the same time. In such cases, SUBF might be a better method than MUBF with

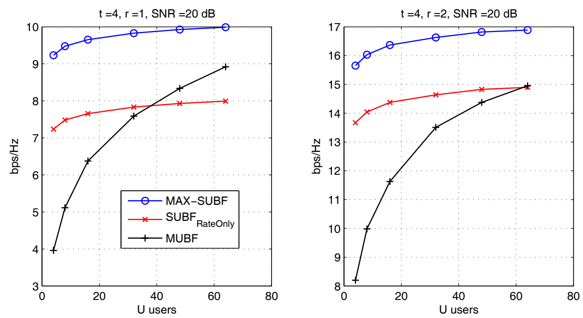

Fig. 3. Ergodic sum rates when $M_T > M_R$ (ρ = 20dB).

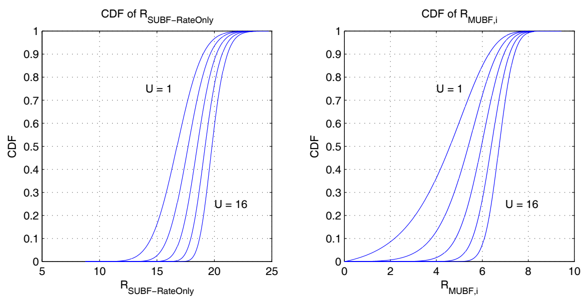

Fig. 4. CDF of $R_{SUBF-RateOnly}$ and $R_{MUBF,i}$ when $M_T = M_R = 3$ and ρ = 20dB.

partial CSI feedback.

On the other hand, Fig. 2 shows the same results in a low SNR region. Note that the gap between the $\mathbb{E}[R_{MAX-SUBF}]$ and $\mathbb{E}[R_{SUBF-RateOnly}]$ is now very large and the power optimization at the transmitter gives R_{SUBF} a noticeable gain over $R_{SUBF-RateOnly}$. Interestingly, in the low SNR region, the threshold does not change much as the number of antennas increases.

When $M_T > M_R$ and ρ is high, MUBF is further limited by the stream interferences. When $M_T = 4$ and $M_R = 1$, for example, the threshold becomes even higher, which is about $U = 39$ in Fig. 3. Since MUBF is meaningful only when $M_T \leq (U \cdot M_R)$, this simulation is performed starting $U = 4$.

VI. Cross-over point approximation

The exact analytic expression for the cross-over point is very difficult to find. This section uses the Gaussian approximation of Section III for $R_{SUBF-RateOnly}$ and ZF approximation of Section IV for R_{MUBF} in order to find the cross-over point.

Only the case of $M_T = M_R$ is considered in this section. Although (11) gives the variance of $\log_2^2 e$ when $M_T = M_R$, a heuristic adjustment is needed for small M_R, which leads to a variance of $0.85 \times \log_2^2 e$ that fits the Monte Carlo simulation best. Meanwhile, the mean of $R_{u,SUBF-RateOnly}$ is empirically computed.

Fig. 4 illustrates the CDF of $R_{SUBF-RateOnly}$ and $R_{MUBF,i}$ as U varies. Using these CDFs, $\mathbb{E}[R_{SUBF-RateOnly}]$ and $\mathbb{E}[R_{MUBF,i}]$ can be numerically

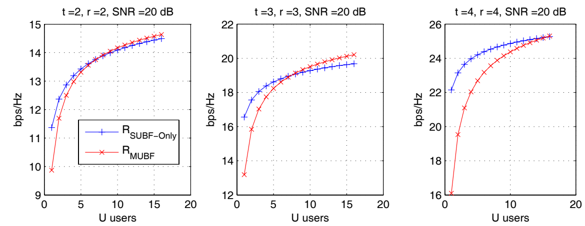

Fig. 5. Cross-over points obtained by approximated distributions.

calculated. The symmetric distribution of H_u makes the average rate of every post-equalized stream equal, which makes $\mathbb{E}[R_{MUBF}] = M_T \cdot \mathbb{E}[R_{MUBF,i}]$.

Fig. 5 illustrates the plots obtained using these approximations. Note that they resemble the Monte Carlo simulation results in Fig. 1 very well.

VII. Conclusion

This paper considers transmitting schemes in MIMO BC with partial CSI feedback constraints. MUBF serves more than one user at a time and asymptotically performs equally with the sum capacity achieving scheme, i.e., DPC. However, it suffers from SINR loss due to interferences among data streams. More streams and high SNR make these interferences even higher and make MUBF unfavorable. MUBF also requires synchronization among the active users, which is challenging in practice. Instead, SUBF serves only one user at a time and outperforms MUBF in terms of sum-rate for the number of active users less than a certain threshold.

References

[1] P. Viswanath, D. N. C. Tse, "Sum capacity of the vector Gaussian broadcast channel and uplink-downlink duality," *Information Theory, IEEE Transactions*, Vol. 49 , Issue: 10 , Aug. 2003, pp.1912-1921.

[2] G. Caire and S. Shamai, "On the achievable throughput of a multiantenna Gaussian broadcast channel," *Information Theory, IEEE Transactions*, Vol. 49 , Issue: 7 , July 2003, pp.1691-1706.

[3] Y. Kim, K. B. Song, R. Narasimhan and J. M. Cioffi, "Single user random beamforming in Gaussian MIMO broadcast channels," *Communications, IEEE International Conference on,* Vol. 4, May 2005, pp.2695-2699.

[4] P. Viswananth, D. Tse and R. Laroia, "Opportunistic beamforming using dumb antennas," *IEEE Trans. Inform. Theory*, vol. 48, June 2002, pp. 1277-1294.

[5] D. N. C. Tse, "Optimal power allocation over parallel Gaussian channels," in *Proc. Int. Symp. Information Theory*, Ulm, Germany, June 1997.

[6] W. Yu, "A Dual Decomposition Approach to the Sum Power Gaussian Vector Multiple Access Channel Sum Capacity Problem", *37th Annual Conference on Information Sciences and Systems (CISS)*, March 2003.

[7] B. M. Hochwald, T. L. Marzetta and V. Tarokh, "Multiple-antenna channel hardening and its implications for rate feedback and scheduling," *Information Theory, IEEE Transactions*, Vol. 50, Issue: 9, Sept. 2004, pp.1893-1909

[8] J. M. Cioffi, *EE 379C Class notes*, Stanford University, available at *http://www.stanford.edu/class/ee379c*

[9] A. Paulraj, R. Nabar and D. Gore, *Introduction to Space-Time Wireless Communications,* 1st Ed. New York: Cambridge University Press, 2003.

[10] M. Sharif and B. Hassibi, "On the capacity of MIMO broadcast channels with partial channel state information," *Information Theory, IEEE Transactions*, Vol. 51, Issue 2, Feb. 2005, pp. 506-522