AN EFFICIENT ALGORITHM FOR OPTIMUM POWER ALLOCATION IN A DECODE-AND-FORWARD COOPERATIVE SYSTEM WITH ORTHOGONAL TRANSMISSIONS

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ABSTRACT

We propose an efficient algorithm that computes the maximum achievable rate of a decode-and-forward multi-relay cooperative system with orthogonal transmissions under total power constraint. In [1] we have argued that in a system with Q relays the optimum rate can be found by solving Q convex problem despite the fact that the optimization problem is not convex and it belongs to a class of problems with variational inequality constraints. In this paper we show that the number of computations can be reduced from solving Q convex problem to solving only $\lceil \log(Q) \rceil + 1$ convex problems. The proposed algorithm facilitates comparisons with competing alternatives in cooperative systems such as the amplify-and-forward or the compress-and-forward strategies.

1. INTRODUCTION

The capacity of the Gaussian relay channel remains an unknown even though considerable progress has been made since the seminal work by Cover and El Gamal in [2], which by establishing essential capacity bounds in the relay channel opens the door for network information theory. The work in [2] has also brought forward two relaying strategies: the decode-and-forward (DF) strategy in which the relay is required to perfectly decode the information symbols received from the source before forwarding them to the destination and the compress-and-forward (CF) strategy in which the relay sends a compressed version of the received symbols to the destination. The later approach has been also used by [3] to establish the ergodic capacity of the flat fading relay channel when the relay is close enough to the destination. In addition, [3] generalizes the bounds of [2] to the multi-relay channel. Other notable information theoretic results in the multi-relay channel could be found in [4–8].

In this paper we depart from the classical multi-relay setup, where the cooperative radios transmit using the same channel resources, by considering orthogonal transmissions from the Q relays available in the system. Using frequency division it is possible to split the channel from the source to the relays and the receiver into $Q + 1$ frequency bands or similarly split the channel to the receiver from the relays and the source. The first approach has been analyzed in [9], where it has been shown that the generalized block Markov coding strategy of [2] achieves capacity for $Q = 1$. Because for the second approach, which is the focus of this paper, the capacity achieving relaying strategy is unknown, we suggest computing the maximum achievable rate under the constraint that relays are required to use a DF modus operandi. The problem formulation is similar to [10, 11], i.e, the achievable rate of a cooperative system with orthogonal transmissions from the relays is maximize under a total power constraint. However, the optimum rate/power allocation results in both [10] and [11] are obtained in a wideband orthogonal DF system derived by letting the channel bandwidth go to infinity. This approach yields a solution where no more than one relay is allowed to transmit at one moment in time. In this paper we analyze the case of finite channel bandwidth and show that the optimum power allocation, which can be found in polynomial time, is a water-filling solution given the set of active relays. In addition, we improve on the algorithm proposed in [1] by showing that the global optimum can be found by solving only $\lceil \log(Q) \rceil + 1$ convex problems, $\lceil \cdot \rceil$ denotes the integer ceil operation.

2. SYSTEM MODEL

We consider a wireless communication system where Q intermediary relays, $\{\mathcal{R}_q\}_{q=1}^{Q}$ help the source communicate with the destination. The Q relays decode the information symbols received from the source and forward the decoded symbols to the destination. The source and the relays use orthogonal transmissions, such as time or frequency division multiplexing (TDM or FDM), to communicate interference-free with the destination. The TDM or the FDM cooperative scheme could be envisioned as an initial upgrade to current cellular systems using time or frequency division multiple access, where idled users situated in the vicinity of an active user would operate as relays. In addition, the orthogonality of the transmitted signals enforces the half-duplex constraint and eliminates possible inter-relay interference.

A relay discovery protocol, which is not the topic of this paper, is run regularly to insure that all the users have discovered their peers. The protocol also updates the user channel state information matrix, which contains information about the quality of all the communication paths in the system. Before we proceed with the channel model description, we simplify the exposition by focusing only on the information symbol transmitted by the source at time i since with orthogonal transmissions the signals received by the destination are not affected by relay induced inter-symbol interference. The information symbol x_s, broadcasted by the source with average power ε_0, is received both at the relays and at the destination. The relays reconstruct the sequence of symbols transmitted by the source. The sequence is re-encoded and forwarded to the destination as it is illustrated by multi-relay channel model depicted in Fig. 1.

Under the assumption that all links in Fig. 1 are affected by multiplicative fading, we can describe the cooperative system using the input-output equation for the one-hop propagation between the source and the destination, i.e.,

$$y_{d_0} = h_{sd} x_s + z_{d_0}, \tag{1}$$

and the equations for the Q parallel two-hop links via the relays

$$\begin{aligned} y_{r_q} &= h_{sr_q} x_s + z_{r_q} \\ y_{d_q} &= h_{r_q d} x_q + z_{d_q}, \quad q \in \{1, ..., Q\}, \end{aligned} \tag{2}$$

where the average power of the symbol transmitted by the relay $\mathcal{R}_q$ is $\varepsilon_q = \mathrm{E}[|x_q|^2]$, $q \in \{1, ..., Q\}$. We assume that the channel between the source and the destination, h_{sd}, the channels between source and the relays, i.e., $\{h_{sr_q}\}_{q=1}^{Q}$, and the channels between the relays and the destination, $\left\{h_{r_q d}\right\}_{q=1}^{Q}$, are fixed (time-invariant) and known to all terminals in the network. We also assume that the additive noises $\left\{z_{r_q}\right\}_{q=0}^{Q}$, $\left\{z_{d_q}\right\}_{q=1}^{Q}$ are independent and complex Gaussian-distributed with independent and identical distributed real and imaginary components. To account for possibly different noise powers at the relays and the destination, we take the variance of z_{r_q} to be $N_r/2$ per dimension for any $q \in \{1, ..., Q\}$, and the variance of z_{d_q} to be $N_0/2$ per dimension for any $q \in \{0, ..., Q\}$

Having introduced the system model, we can informally state the problem. Given that *the relays decode perfectly* the transmissions from the source, we want to find the maximum achievable rate between the source and the destination subject to the power constraint

$$\boldsymbol{\varepsilon}^T \boldsymbol{\varepsilon} \leqslant \mathcal{P}, \tag{3}$$

where $\boldsymbol{\varepsilon} := [\varepsilon_0, \varepsilon_1, ..., \varepsilon_Q]^T$. One may argue at this point that, instead of a total power constraint, individual power constraints should be set for each relay since the relays use separate batteries. Nevertheless, it is possible to show that, given a functional symmetry in the network, (i.e., as time evolves each relay becomes a source and vice-versa) optimizing power over many symbols, i.e., across time, for each relay is equivalent to optimizing power per symbol across space, i.e., among

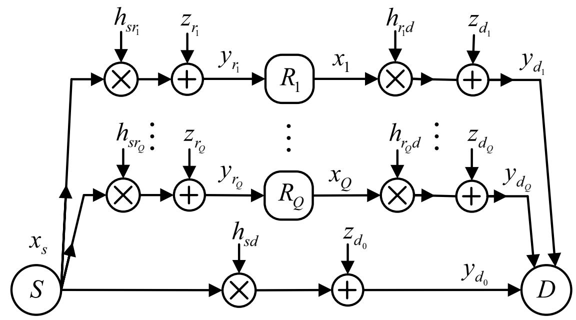

Fig. 1. The multi-relay channel with orthogonal transmissions.

relays with a total power budget as in (3). Before we can mathematically formulate the optimization problem, we need to elaborate on the proposed setup.

We assume that there is no collaboration between relays, i.e., relays do not acquire any information about the signals received from the source by the other relays. Any *active* relay $\mathcal{R}_q$ (i.e., any $\mathcal{R}_q$ for which $\varepsilon_q > 0$) should be able to decode the message x_s broadcasted by the source. This is possible if

$$R < I(x_s; y_{r_q}), \quad q \in A_Q, \tag{4}$$

where $A_Q \subseteq \{1, ..., Q\}$ is the index set for the active relays. In order to perfectly decode at the destination the orthogonal transmissions sent by the source and the relays, it is required that in addition to (4)

$$R < I(x_s, \mathbf{x}_r; \mathbf{y}_d), \tag{5}$$

where $\mathbf{x}_r := [x_1, \ldots, x_Q]$ and $\mathbf{y}_d := [y_{d_0}, y_{d_1}, ..., y_{d_Q}]^T$. From (4) and (5) we find that the maximum achievable rate for the multi-relay channel with perfect decoding at the relays is

$$R^* = \max_{p(x_s, \mathbf{x}_r)} \min \left\{ \{I(x_s; y_{r_q})\}_{q \in A_Q^*}, I(x_s, \mathbf{x}_r; \mathbf{y}_d) \right\}. \tag{6}$$

where A_Q^* is an *optimum* index set for the active relays.

Considering that the transmitted signals propagate through white Gaussian noise links, and are subject to the total average energy per symbol constraint in (3), we specialize the mutual information in (4) and (5) to obtain the achievable rate

$$R^* = \max_{\varepsilon_0 + \sum_{q \in A_Q^*} \varepsilon_q \leqslant \mathcal{P}} \min \left\{ \{R_q(\boldsymbol{\varepsilon})\}_{q \in A_Q^*}, R_s(\boldsymbol{\varepsilon}) \right\}, \tag{7}$$

where

$$\begin{aligned} R_q(\boldsymbol{\varepsilon}) &:= \log(1 + \gamma_{sr_q} \varepsilon_0), \quad q \in A_Q^*, \\ R_s(\boldsymbol{\varepsilon}) &:= \log(1 + \gamma_0 \varepsilon_0) + \sum_{q \in A_Q^*} \log(1 + \gamma_q \varepsilon_q), \end{aligned}$$

$\gamma_{sr_q} := |h_{sr_q}|^2 / N_r$, $\gamma_0 := |h_{sd}|^2 / N_0$, and $\gamma_q := |h_{r_q d}|^2 / N_0$, $q \in \{1, ..., Q\}$. In the next section we present an efficient approach for solving the problem in (7).

3. AN EFFICIENT ALGORITHM FOR OPTIMUM POWER ALLOCATION

Before describing the algorithm, we borrow a lemma from [1] that could reduce the size of the optimization problem in (7) and for completeness we also outline its proof.

Lemma 1. *If there exists a set of relays* $\{\mathcal{R}_i \,|\, i \in I_Q \subset \{1,...,Q\}\}$ *such that* $\gamma_{sr_i} \leqslant \gamma_0$ *for* $i \in I_Q$*, then the relays* $\{\mathcal{R}_i \,|\, i \in I_Q\}$ *are not part of the optimum set of active relays, i.e.,* $I_Q \subseteq {A_Q^*}^c$*, where* ${A_Q^*}^c$ *is the complement of* A_Q^**.*

Proof. The proof is by contradiction. Let us assume that I_Q and A_Q^* are not disjoint. Hence, there exist a non-empty set $I'_Q \subseteq I_Q$ such that $I'_Q \subseteq A_Q^*$. Let us select $i' \in I'_Q$ and because relay $\mathcal{R}_{i'}$ is active, $R^* < \log(1+\gamma_{sr_{i'}}\varepsilon_0)$. However, if all relays $\mathcal{R}_i$ with $i \in I'_Q$ are powered off, then $R^* > \log(1+\gamma_0\varepsilon_0)$. In the statement of Lemma 1 we assume $\gamma_0 \geqslant \gamma_{sr_{i'}}$, and therefore, $R^* > \log(1+\gamma_0\varepsilon_0) \geqslant \log(1+\gamma_{sr_{i'}}\varepsilon_0) > R^*$, which is a contradiction. □

As observed in [1], Lemma 1 suggests that a simplification of the call setup procedure for the cooperative system can be obtained by excluding distant relays.

The epigraphic form of the optimization problem in (7) is

$$\begin{aligned} &\text{minimize} - R \\ &\text{subject to} \\ &\qquad R \leqslant \log(1+\gamma_0\varepsilon_0) + \sum_{q=1}^{Q}\log(1+\gamma_q\varepsilon_q) \\ &\qquad 1(\varepsilon_q)R \leqslant \log(1+\gamma_{sr_q}\varepsilon_0), \quad q \in \{1,...,Q\} \\ &\qquad \sum_{q=0}^{Q}\varepsilon_q = \mathcal{P} \text{ and } \varepsilon_q \geqslant 0, \quad q \in \{1,...,Q\}, \end{aligned} \tag{8}$$

where $1(\varepsilon_q) = 1$ if $\varepsilon_q > 0$ and $1(\varepsilon_q) = 0$ if $\varepsilon_q = 0$ is an indicator function for the activity of relay R_q.

The problem in (8) is not convex. If we rewrite the inequalities $1(\varepsilon_q)R \leqslant \log(1+\gamma_{sr_q}\varepsilon_0)$, $q \in \{1,...,Q\}$, in (8) as $\varepsilon_q\left(R - \log(1+\gamma_{sr_q}\varepsilon_0)\right) \leqslant 0$, $q \in \{1,...,Q\}$, we can easily see that (8) belongs to the class of problems with variational inequality constraints, which in general are difficult to solve [12].

The difficulty in (7) (or (8)) comes from trying to establish the optimum set of active relays, i.e., A_Q^*. If we fix the set of active relays, i.e., fix $A_Q = \tilde{A}_Q$, then (8) becomes

$$\begin{aligned} &\text{minimize} \; - R \\ &\text{subject to} \quad R \leqslant \log(1+\gamma_0\varepsilon_0) + \sum_{q\in\tilde{A}_Q}\log(1+\gamma_q\varepsilon_q) \\ &\qquad\qquad R \leqslant \log(1+\gamma_{sr_q}\varepsilon_0), \quad q \in \tilde{A}_Q \\ &\qquad\qquad \sum_{q\in\tilde{A}_Q}\varepsilon_q = \mathcal{P} \text{ and } \varepsilon_q \geqslant 0, \quad q \in \tilde{A}_Q, \end{aligned} \tag{9}$$

which is a convex problem. Since there are 2^Q possible A_Qs, one actually needs to solve 2^Q convex problems, which yield the rates $\{R_p^*\}_{p=1}^{2^Q}$. The maximum rate, R^*, and correspondingly the optimum power allocation, could be found by taking the maximum over all R_p^*s. Of course, we do not advocate solving 2^Q convex problems, since the complexity of the algorithm increases exponentially with the size of the problem. In [1] we have solved a similar problem and argued that a solution to (7) can be found by solving Q convex problems. We improve on our previous result in the following theorem.

Theorem 1. *If* $\gamma_{sr_i} \neq \gamma_{sr_j}$ *for any* $i \neq j$*, with* $i, j \in \{1,...,Q\}$*, then the solution to the problem in* (7) *can be found by solving at most* $\lceil\log_2(Q)\rceil + 1$ *convex problems.*

Proof. The proof is based on observing that the set of inequalities $1(\varepsilon_q)R \leqslant \log(1+\gamma_{sr_q}\varepsilon_0)$, $q \in \{1,...,Q\}$ in (8) can be reduced to one inequality. Without loss of generality, let us assume that $\gamma_{sr_{q+1}} \geqslant \gamma_{sr_q}$ for $q \in \{1,...,Q-1\}$. Using the monotonicity of the logarithm, we can write $\log(1+\gamma_{sr_{q+1}}) \geqslant \log(1+\gamma_{sr_q})$ for $q \in \{1,...,Q-1\}$, and replace the set of inequalities

$$1(\varepsilon_q)R \leqslant \log(1+\gamma_{sr_q}\varepsilon_0), \;\; q \in \{1,...,Q\}, \tag{10}$$

in (8), which introduce problematic complementary conditions, with

$$R \leqslant \log(1+\gamma_{sr_{q^*}}\varepsilon_0), \tag{11}$$

where q^* is the minimum relay index in the optimum set of active relays, i.e., $q^* = \min\{q \,|\, q \in A_Q^*\}$. We do not know the optimum set of active relays, but we know that $q^* \in \{1,...,Q\}$. So instead of solving (9) for all choices of A_Q, we can solve the convex problem

$$\begin{aligned} &\text{minimize} - R \\ &\text{subject to} \\ &\qquad R \leqslant \log(1+\gamma_0\varepsilon_0) + \sum_{q=\tilde{q}}^{Q}\log(1+\gamma_q\varepsilon_q) \\ &\qquad R \leqslant \log(1+\gamma_{sr_{\tilde{q}}}\varepsilon_0) \\ &\qquad \sum_{q=\tilde{q}}^{Q}\varepsilon_q = \mathcal{P} \text{ and } \varepsilon_q \geqslant 0, \quad q \geqslant \tilde{q}, \end{aligned} \tag{12}$$

for all choices of $\tilde{q}$. For each $\tilde{q} \in \{1,...,Q\}$ we obtain a rate $R_{\tilde{q}}^*$ by solving (12). The optimum rate $R^* = \max\{\tilde{q} \in \{1,...,Q\}|\, R_{\tilde{q}}^*\}$, requires solving (12) Q-times.

We want to point out that as $\tilde{q}$ varies, the two constraints on rate R in (12) move in opposite directions. For example, as $\tilde{q}$ increases, $\log(1+\gamma_0\varepsilon_0) + \sum_{q=\tilde{q}}^{Q}\log(1+\gamma_q\varepsilon_q)$ decreases, while $\log(1+\gamma_{sr_{\tilde{q}}}\varepsilon_0)$ increases. Consequently, when $\tilde{q}$ increases, the behavior of the dominant constraint on R (i.e., the strongest of the two constraints in (12)) can be split in 3 successive stages, which may not all take place: *stage 1)* becomes more relaxed; *stage 2)* does not change; *stage 3)* becomes tighter.

Therefore, as $\tilde{q}$ increases, $R^*_{\tilde{q}}$, which is a function of $\tilde{q}$, increases, becomes constant, and then decreases, where any one or two of these stages may not take place. If we extend the definition of strictly quasi-convexity to functions with a discrete domain, we conclude that $R^*_{\tilde{q}}$ is a strictly quasi-concave function of $\tilde{q}$. Note that without $\gamma_{sr_i} \neq \gamma_{sr_j}$ in the theorem's statement, we can only guarantee quasi-concavity, but not strict quasi-concavity. There is no need to enumerate all $\tilde{q}$ as we have done previously since we only need to look for any maximum $R^*_{\tilde{q}}$ as a function of $\tilde{q}$. This can be done using bisection and running (12) only $(\lceil \log(Q) \rceil + 1)$-times. $\square$

Algorithm. *Solve* (12) *for $\tilde{q}$ at midpoint, i.e., $\tilde{q} = \tilde{q}_{\text{mid}} := \lceil Q/2 \rceil$ and for $\tilde{q}$ at midpoint minus one, i.e., $\tilde{q} = \tilde{q}_{\text{mid}} - 1$. Compare $R_{\tilde{q}_{\text{mid}}}$ and $R_{\tilde{q}_{\text{mid}}-1}$. If $R_{\tilde{q}_{\text{mid}}} > R_{\tilde{q}_{\text{mid}}-1}$, then an optimum $\tilde{q} = q^*$ is in the set $\{\tilde{q}_{\text{mid}}, ..., Q\}$. Otherwise, q^* is in the set $\{0, ..., \tilde{q}_{\text{mid}} - 1\}$. At each iteration the algorithm produces an interval that contains q^*, but it is half the size of the original interval. The algorithm ends when an optimum q is located.*

From (12) we can easily remark that given the optimum set of active relays the solution follows the water-filling principle. It can also be easily generalized to the bandwidth constraint problem. Given the bandwidth set $\{W_0, W_1, ..., W_Q\}$ for the FDM channel, one needs to replace the rate constraints in (8) with $R \leqslant W_0 \log(1 + \gamma_0 \varepsilon_0 / W_0) + \sum_{q=1}^{Q} W_q \log(1 + \gamma_q \varepsilon_q / W_q)$, $1(\varepsilon_q) R \leqslant W_q \log(1 + \gamma_{sr_q} \varepsilon_0 / W_q)$ and follow an approach similar to the one described in the proof of Theorem 1.

Simulations (see [1]) suggest that only a few relays are required to achieve most of the rate advantage provided by the cooperative setup. Similar to multi-relay systems with interfering transmissions, the decode-and-forward scheme with orthogonal transmissions yields a higher rate than its amplify-and-forward (AF) counterpart if the relays are sufficiently close to the source. Unlike the AF scheme, which does not benefit significantly from the optimization process, assigning the optimum powers to the regenerative relays leads to a considerable rate increase for the DF scheme when compared to the equal power allocation setup.

4. CONCLUSIONS

In this paper we have focused on a decode-and-forward cooperative system with orthogonal transmissions and we have analyzed its achievable rate for the case when the relays do not collaborate with each other. We have shown that the maximum achievable rate for a system with non-collaborative relays can be found in log-polynomial time despite the fact that the optimization problem is not convex and it belongs to a class of problems with variational inequality constraints. Preliminary simulations show that DF benefits considerably from the optimization process. However, using more than a couple of relays brings a minimal rate advantage for the DF setup.

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