CLOSED-FORM ESTIMATORS OF CARRIER FREQUENCY OFFSETS AND CHANNELS IN THE UPLINK OF MULTIUSER OFDM SYSTEMS

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ABSTRACT

This contribution proposes a new joint estimation method of multiple carrier frequency offsets (CFOs) and channels in the uplink of multiuser orthogonal frequency division multiplexing (OFDM) systems. The estimators are derived from the optimal maximum-likelihood (ML) principle. Complexity reductions are achieved by exploiting the correlation properties of the training-sequence. The grid search algorithm is converted into a polynomial root finding procedure which leads to closed-form estimators for moderate CFOs. Numerical results confirm that the performance degradation due the approximations compared to the Cramer-Rao bound (CRB) is small and may be negligible in practice.

1. INTRODUCTION

Besides its advantages of high spectral efficiency and low-complexity receiver structures, OFDM is known to have some major drawbacks like a highly increased sensitivity to CFOs caused by oscillator mismatches as well as motion-induced Doppler shifts. Synchronization in frequency must therefore be counteracted to avoid severe error-rate degradations. However, in the uplink of multiuser systems, the estimation of multiple CFOs is strongly coupled with the task of channel estimation which makes the optimal joint maximum-likelihood solution unfeasible in practical implementations.

Whereas a huge amount of literature about synchronization in single-user OFDM systems exists (see e.g. [1]), only a few contributions have addressed this uplink synchronization problem in multiuser OFDM, e.g. [2, 3]. However, the proposed estimators are designed for particular subcarrier allocations as given by the orthogonal frequency division multiple access (OFDMA) scheme.

Recently, the authors in [4] proposed an approximative ML estimation method of multiple frequency offsets and channel impulse responses by using distinct correlation properties of the training signal. But still an extensive grid search must be carried out. In this contribution, we extent this method by converting the nonlinear cost function into a polynomial series from which the maximum can be found more easily. For moderate CFOs, even a closed-form estimator is given. Hence, this contribution fills the gap well to [5], where we have presented a fine synchronization technique (based on pilots in the frequency domain) which requires a reliable initial channel estimate.

The paper is organized as follows: In section 2, an uplink multiuser system model including user-dependent CFOs is introduced. Based on the correlation properties of the training sequence, low complexity approximative ML estimators of the user CFOs and channel impulse responses (CIRs) are derived in section 3. Section 4 analyzes the estimation performance in terms of the CRB. In section 5, numerical results are shown in order to validate the high estimation accuracy. Finally, conclusions are drawn in section 6.

2. SYSTEM MODEL

We consider the uplink of a general multiuser OFDM system with P users equipped with one transmit antenna and a base station with Q receive antennas. Due to the simultaneous access of all users, the received time-discrete baseband signal at the q-th base station antenna is

$$r_i^{(q)} = \sum_{p=1}^{P} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N} f_\varepsilon^{(p)} i} \sum_{l=0}^{L-1} x_{i-l}^{(p)}\, h_l^{(qp)} + v_i^{(q)} \,, \tag{1}$$

where $x_i^{(p)}$ is the i-th sample of the transmitted signal of the p-th user and $h_l^{(qp)}$ denotes the l-th path coeffient of the time-invariant propagation channel between user p and the q-th receive antenna, including transmit and receive filters. The (normalized) carrier frequency offsets is $f_\varepsilon^{(p)} = \Delta f^{(p)} NT$ (with absolute frequency offsets $\Delta f^{(p)}$, fast Fourier transform (FFT) length N and sampling period T), $v_i^{(q)}$ is complex white Gaussian (thermal) noise filtered by the receive filter.

Since the estimation method operates on one training block of length N in the time-domain, its field of application is not restricted to OFDM. By collecting N samples in $\mathbf{r}^{(q)} = [r_0^{(q)}, \ldots, r_{N-1}^{(q)}]^{\mathrm{T}}$, (1) can be formulated in matrix-vector notation as

$$\mathbf{r}^{(q)} = \sum_{p=1}^{P} \mathbf{C}_{\varepsilon}^{(p)} \tilde{\mathbf{A}}^{(p)} \mathbf{h}^{(qp)} + \mathbf{v}^{(q)} \tag{2}$$

$$= \breve{\mathbf{A}}(\mathbf{f}_{\varepsilon}) \mathbf{h}^{(q)} + \mathbf{v}^{(q)} \tag{3}$$

with the CFO matrix $\mathbf{C}_{\varepsilon}^{(p)} = \mathrm{diag}(1, \ldots, \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}(N-1)f_{\varepsilon}^{(p)}})$, the channel vector $\mathbf{h}^{(q)} = [h_0^{(q1)}, \ldots, h_{L-1}^{(q1)}, \ldots, h_{L-1}^{(qP)}]^{\mathrm{T}}$ and

$$\breve{\mathbf{A}}(\mathbf{f}_{\varepsilon}) = \left[\mathbf{C}_{\varepsilon}^{(1)} \tilde{\mathbf{A}}^{(1)}, \mathbf{C}_{\varepsilon}^{(2)} \tilde{\mathbf{A}}^{(2)}, \ldots, \mathbf{C}_{\varepsilon}^{(P)} \tilde{\mathbf{A}}^{(P)} \right] . \tag{4}$$

The signal matrix $\tilde{\mathbf{A}}^{(p)}$ is

$$\tilde{\mathbf{A}}^{(p)} = \begin{bmatrix} x_0^{(p)} & \ldots & x_{-L+1}^{(p)} \\ \vdots & \ddots & \vdots \\ x_{N-1}^{(p)} & \ldots & x_{-L+N}^{(p)} \end{bmatrix} \tag{5}$$

considering that a cyclic prefix of length $N_{\mathrm{g}} \geq L$ preceeds the training block. Stacking signals from all antennas together, the resulting signal model takes the form

$$\mathbf{r} = \mathbf{A}(\mathbf{f}_{\varepsilon}) \mathbf{h} + \mathbf{v} \tag{6}$$

with all received symbols $\mathbf{r} = [(\mathbf{r}^{(1)})^{\mathrm{T}}, \ldots, (\mathbf{r}^{(Q)})^{\mathrm{T}}]^{\mathrm{T}}$, channel coefficients $\mathbf{h} = [(\mathbf{h}^{(1)})^{\mathrm{T}}, \ldots, (\mathbf{h}^{(Q)})^{\mathrm{T}}]^{\mathrm{T}}$ and $\mathbf{A}(\mathbf{f}_{\varepsilon}) = \mathrm{diag}(\breve{\mathbf{A}}(\mathbf{f}_{\varepsilon}), \ldots, \breve{\mathbf{A}}(\mathbf{f}_{\varepsilon}))$.

3. ALGORITHM DESCRIPTION

3.1. Maximum-Likehood approach

As the noise is uncorrelated and complex Gaussian with $\mathcal{N}(\mathbf{0}, \sigma_v^2 \mathbf{I})$, we can apply the maximum-likelihood principle with the log-likelihood function

$$\ln p(\mathbf{r}; \mathbf{f}_{\varepsilon}, \mathbf{h}) = c - \frac{1}{\sigma_v^2} \left[\mathbf{r} - \mathbf{A}\left(\mathbf{f}_{\varepsilon}\right) \mathbf{h} \right]^{\mathrm{H}} \left[\mathbf{r} - \mathbf{A}(\mathbf{f}_{\varepsilon}) \mathbf{h} \right] \, , \tag{7}$$

where $c = -NQ \ln\left(\pi \sigma_v^2\right)$ is a constant. A channel estimator can be found with the aid of the separability property. Taking the derivative with respect to $\mathbf{h}$ and setting it to zero yields

$$\hat{\mathbf{h}} = \left[\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon}) \right]^{-1} \mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{r} \, . \tag{8}$$

Inserting $\hat{\mathbf{h}}$ into (7) and neglecting irrelevant terms leads to the CFO estimator

$$\hat{\mathbf{f}}_{\varepsilon} = \arg\max_{\mathbf{f}_{\varepsilon}} J(\mathbf{f}_{\varepsilon}) \qquad \text{with the cost function}$$

$$J(\mathbf{f}_{\varepsilon}) = \mathbf{r}^{\mathrm{H}} \mathbf{A}(\mathbf{f}_{\varepsilon}) \left[\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon}) \right]^{-1} \mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{r} \, , \tag{9}$$

which takes the channel implicitly into account. Once we have found the frequency offsets, (8) can be applied to estimate the channel vector $\mathbf{h}$. Even for a base station, this multidimensional search is much too complex, especially because of the matrix inversion. Simplifications arise for special structures of the hermitian matrix $\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon})$. This matrix can be partioned into $P \times P$ submatrices

$$\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon}) = \begin{bmatrix} \mathbf{B}^{(11)} & \ldots & \mathbf{B}^{(1P)} \\ \vdots & \ddots & \vdots \\ \mathbf{B}^{(P1)} & \ldots & \mathbf{B}^{(PP)} \end{bmatrix} \tag{10}$$

whose elements are given by

$$\left[\mathbf{B}^{(pp')} \right]_{l,k} = \sum_{n=0}^{N-1} x_{n-l}^{(p)*} x_{n-k}^{(p')} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N} n (f_{\varepsilon}^{(p)} - f_{\varepsilon}^{(p')})} \, . \tag{11}$$

Choosing (shift)-orthogonal training sequences with correlation-properties $\sum_{n=0}^{N-1} x_{n-l}^{(p)*} x_{n-k}^{(p')} = N \delta_{\mathrm{k}}(l-k) \delta_{\mathrm{k}}(p-p')$ (see e.g. CHU-codes), the matrix $\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon})$ can be approximated for small ratios PL/N by

$$\mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{A}(\mathbf{f}_{\varepsilon}) \approx N \mathbf{I}_{PN \times PN} \tag{12}$$

which simplifies the cost function in (9):

$$J(\mathbf{f}_{\varepsilon}) \approx \frac{1}{N} \mathbf{r}^{\mathrm{H}} \mathbf{A}(\mathbf{f}_{\varepsilon}) \mathbf{A}^{\mathrm{H}}(\mathbf{f}_{\varepsilon}) \mathbf{r}$$

$$= \sum_{p=1}^{P} \sum_{q=1}^{Q} \sum_{l=0}^{L-1} \left| \sum_{i=0}^{N-1} r_i^{(q)*} x_{-l+i}^{(p)} \mathrm{e}^{\mathrm{j}\frac{2\pi}{N} f_{\varepsilon}^{(p)} i} \right|^2 . \tag{13}$$

It can be seen that the joint P-dimensional search can be splitted into P independent searches [4]. However, the complexity of grid search algorithms is still quite high which motivates to look for easier methods.

3.2. Polynomial series

The proposed approach is based on the observation that the frequency offsets characterizing matrix $\mathbf{C}_{\varepsilon}^{(p)}$ can be approximated by

$$\mathbf{C}_{\varepsilon}^{(p)} \approx \mathbf{I} + \mathbf{D} f_{\varepsilon}^{(p)} + \frac{1}{2!} \mathbf{D}^2 f_{\varepsilon}^{(p)2} + \ldots + \frac{1}{K'!} \mathbf{D}^{K'} f_{\varepsilon}^{(p)K'}$$

$$= \sum_{\kappa=0}^{K'} \frac{1}{\kappa!} \cdot \mathbf{D}^{\kappa} f_{\varepsilon}^{(p)\kappa} \, . \tag{14}$$

The matrix $\mathbf{D}^{\kappa}$ is defined as

$$\mathbf{D}^{\kappa} = \mathrm{diag}\left(\left(\mathrm{j}\frac{2\pi}{N} \right)^{\kappa} [0^{\kappa}, \ldots, (N-1)^{\kappa}] \right) \, . \tag{15}$$

Therefore, the cost function in (13) (neglecting the irrelevant factor N) for the p-th frequency offset is given by

$$J(f_{\varepsilon}^{(p)}) = \mathbf{r}^{\mathrm{H}} \mathbf{C}_{\varepsilon}^{(p)} \tilde{\mathbf{A}}^{(p)} \tilde{\mathbf{A}}^{(p)\mathrm{H}} \mathbf{C}_{\varepsilon}^{(p)} \mathbf{r}$$

$$= \mathbf{r}^{\mathrm{H}} \left(\sum_{\kappa=0}^{K'} \frac{1}{\kappa!} \mathbf{D}^{\kappa} f_{\varepsilon}^{(p)\kappa} \right) \tilde{\mathbf{A}}^{(p)}$$

$$\cdot \tilde{\mathbf{A}}^{(p)\mathrm{H}} \left(\sum_{\kappa'=0}^{K'} \frac{1}{\kappa'!} \left(\mathbf{D}^{\kappa'} \right)^{\mathrm{H}} f_{\varepsilon}^{(p)\kappa'} \right) \mathbf{r} \, . \tag{16}$$

It can be seen that the cost function is equal to the polynomial

$$J(f_\varepsilon^{(p)}) = \sum_{\kappa=0}^{K} f_\varepsilon^{(p)\kappa} a_\kappa \tag{17}$$

with K denoting the degree of the considered polynomial and

$$a_\kappa = \frac{1}{\kappa!} \sum_{i=0}^{\kappa} \begin{pmatrix} \kappa \\ i \end{pmatrix} \Re\left\{ \mathbf{r}^{\mathrm{H}} \mathbf{D}^i \mathbf{A}^{(p)} \mathbf{A}^{(p)\mathrm{H}} \left(\mathbf{D}^{\kappa-i}\right)^{\mathrm{H}} \mathbf{r} \right\} . \tag{18}$$

From (17), the first and second derivative can be easily calculated:

$$\frac{\partial J(f_\varepsilon^{(p)})}{\partial f_\varepsilon^{(p)}} = \sum_{\kappa=1}^{K} f_\varepsilon^{(p)\kappa-1} \kappa\, a_\kappa \quad \text{and} \tag{19}$$

$$\frac{\partial^2 J(f_\varepsilon^{(p)})}{\partial f_\varepsilon^{(p)2}} = \sum_{\kappa=2}^{K} f_\varepsilon^{(p)\kappa-2} \kappa(\kappa-1)\, a_\kappa \, . \tag{20}$$

Setting the first derivative to zero, we can estimate the frequency offsets now by finding the roots of the polynomial of degree $K-1$:

$$\frac{\partial J(f_\varepsilon^{(p)})}{\partial f_\varepsilon^{(p)}} = \sum_{\kappa=0}^{K-1} c_\kappa f_\varepsilon^{(p)\kappa} \overset{!}{=} 0 \, , \tag{21}$$

where the coefficients c_κ are

$$c_\kappa = (\kappa+1)\; a_{\kappa+1} \, . \tag{22}$$

The CFO estimate of the p-th user is then given by the real root of (21) with negative second derivative which is closest to zero.

3.3. Estimators in closed-form

One major advantage of this estimator class is its scalability - the degree of the polynomial determines the quality of the approximation. For large CFOs, higher-order polynomials must be used. We will see, that for smaller CFOs, smaller values of K are sufficient, thus leading to closed-form solutions of the estimator (for $K < 6$). For $K = 4$ and $K = 5$, a cubic and quartic equation must be solved (e.g. using CARDANO's formula). For $K = 2$, the estimator of the frequency offsets can easily be written as

$$\hat{f}_{\varepsilon|K=2}^{(p)} = \frac{-\Re\left\{\mathbf{r}^{\mathrm{H}} \mathbf{D} \mathbf{A}^{(p)} \mathbf{A}^{(p)\mathrm{H}} \mathbf{r}\right\}}{\Re\left\{\mathbf{r}^{\mathrm{H}} \left(\mathbf{D}^2 \mathbf{A}^{(p)} \mathbf{A}^{(p)\mathrm{H}} + \mathbf{D} \mathbf{A}^{(p)} \mathbf{A}^{(p)\mathrm{H}} \mathbf{D}^{\mathrm{H}}\right) \mathbf{r}\right\}} \, . \tag{23}$$

The fine synchronization method proposed in [5] represents one application area for the estimation of small CFOs. But if even small frequency offsets are not taken into consideration in the channel estimator, the estimation quality degrades thus leading to also worse estimation results of pilot tracking algorithms which are based upon the channel state information. However, by applying (23) and then (8), a reliable initial channel estimate can be found now.

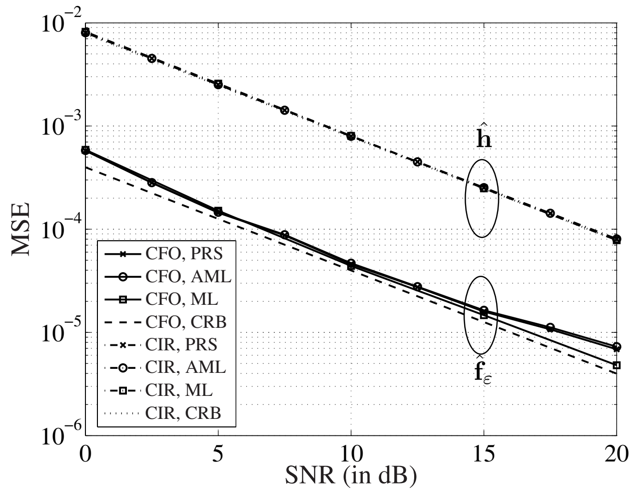

Fig. 1. MSE of $\hat{f}_\varepsilon^{(p)}$ and $\hat{h}_l^{(qp)}$ vs. SNR with different algorithms: (a) ML, (b) AML, (c) PRS with $K = 8$, $f_\varepsilon^{(p)} < 0.5$

4. PERFORMANCE ANALYSIS

The CRB can be calculated by the diagonal elements of the Fisher-information matrix [4]. However, a complicated inverse which depends on the true estimation parameter values must be evaluated. To gain more insight, it can be seen that the CRBs of $\mathbf{f}_\varepsilon$ and $\mathbf{h}$ are well approximated by

$$\mathrm{CRB}(\hat{f}_\varepsilon^{(p)}) \approx \frac{3PN^2}{Q\pi^2 N(N-1)(2N-1)\mathrm{SNR}} \tag{24}$$

$$\mathrm{CRB}(\hat{h}_l^{(qp)}) \approx \frac{P}{N\mathrm{SNR}} \tag{25}$$

with $\sum_{l=0}^{L-1} |h_l^{(qp)}|^2 = 1$ and the signal-to-noise power ratio $\mathrm{SNR} = \mathrm{E}\{|r_i^{(q)}|^2\}/\mathrm{E}\{|v_i^{(q)}|^2\}$. It turns out that the estimation accuracy of the proposed method lies very close to this theoretical limit.

5. NUMERICAL RESULTS

In this section, simulations are carried out in order to validate the performance of the proposed coarse sychronization in terms of the mean-squared error (MSE) of the estimation results. The multipath channel is implemented as a tapped-delay line with Rayleigh fading coefficients and a power delay profile given by the typical indoor models used for IEEE 802.11a. The main system parameters are given by the FFT-size $N = 256$, the guard interval length $N_{\mathrm{g}} = N/8 = L$ and the sampling period $T = 50$ ns. Only one OFDM block as training symbol with cyclic prefix is used. We consider a system with two users ($P = 2$) and a base station with three antennas ($Q = 3$). The user CFOs are treated as uniformly distributed random variables with different maximum values.

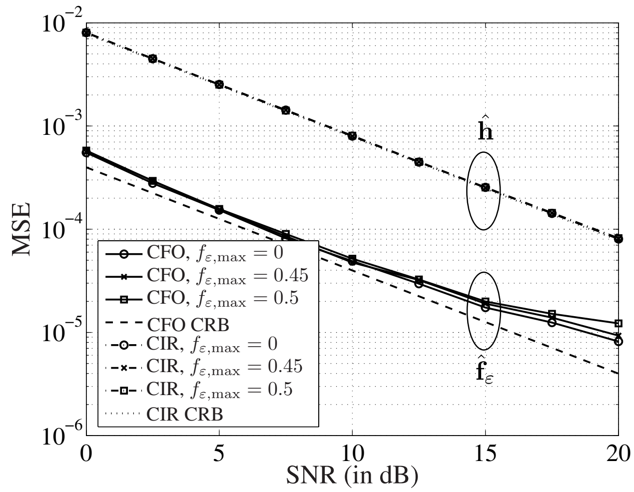

Fig. 2. MSE of $\hat{f}_\varepsilon^{(p)}$ and $\hat{h}_l^{(qp)}$ vs. SNR using PRS algorithm with $K = 5$ and different maximum values of the CFOs

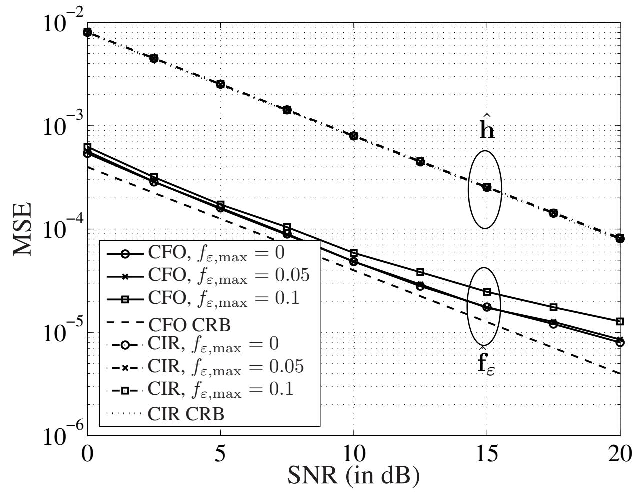

Fig. 3. MSE of $\hat{f}_\varepsilon^{(p)}$ and $\hat{h}_l^{(qp)}$ vs. SNR using PRS algorithm with $K = 2$ and different maximum values of the CFOs

In Fig. 1, the MSE values of the estimated CFOs and channel coefficients versus the SNR for three different algorithms is compared: (a) Optimal ML estimation (9), (b) estimation via the approximated cost function (13) denoted by approximative ML (AML) and (c) estimation via polynomial root search (PRS) of degree $K = 8$. The estimates of $\mathbf{h}$ are found by applying (8). As a reference, the approximated CRBs are depicted. A slight performance degradation can be observed for the CFO MSEs of the AML compared to the ML algorithm because of the approximation in (12). However, for $K = 8$, the performance of the proposed PRS algorithm equals that of the AML algorithm. In order to further decrease the complexity, the degree of the polynomials is reduced such that closed-form expressions can be formulated. Fig. 2 depicts the MSE of $\hat{\mathbf{f}}_\varepsilon$ and $\hat{\mathbf{h}}$ versus SNR for different maximum values $f_{\varepsilon,\max}$ of the CFOs when $K = 5$ is set. Of course, the polynomial approximation gets worse with increasing $f_{\varepsilon,\max}$ and thus the accuracy for larger acquisition ranges decreases. However, the performance degradation of the CFO estimator until $f_{\varepsilon,\max} = 0.5$ is very small, that of the channel estimator not even visible within this SNR range. When using the simple estimator (23), the degradation sets in for $f_{\varepsilon,\max} = 0.1$ which can be seen from Fig. 3. Nevertheless, the proposed joint CFO and channel estimation is suitable as initial stage of pilot based tracking schemes like in [5].

6. CONCLUSION

In this contribution, a novel approximative ML estimation of multiple carrier frequency offsets (CFOs) and channel impulse responses in multiuser systems in application to OFDM has been proposed. The estimator of the CFOs is mainly based upon finding roots of a polynomial, whereas the channel estimator adapts the conventional least-squared approach by taking the influence of the CFOs into account. This enables, for moderate CFOs, to provide attractive closed-form solutions. It turns out that the mean-squared estimation error is close to the CRB within the most interesting SNR region. Due to the reduced complexity at marginally lower accuracy compared the optimal maximum-likelihood estimation, the proposed technique can serve as a suitable initial carrier frequency and channel estimation method in practice.

7. REFERENCES

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