GLOBAL STABILITY OF A POPULATION OF MUTUALLY COUPLED OSCILLATORS REACHING GLOBAL ML ESTIMATE THROUGH A DECENTRALIZED APPROACH

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ABSTRACT

The mathematical models of populations of mutually coupled oscillators having self-synchronization capabilities are a powerful tool for designing sensor networks with high energy efficiency, fault tolerance and scalability. In this work, we derive the conditions for the existence the asymptotic stability of the equilibrium of a system capable to provide maximum likelihood estimates through only local coupling and without the need for a fusion center, provided that the whole network observes the same phenomenon. Interestingly, we show that the network global consensus capability is strictly related to the network topology. Finally we test the performance taking into account propagation delays and possible parameter fluctuations among the network nodes.

1. INTRODUCTION

One of the main problems in current sensor networks research is how to convey the necessary amount of data from the network nodes to a fusion center in the most efficient manner. This entails proper combination of source coding and modulation in order to get the best trade-off in terms of transmission rate and final decision accuracy. Not surprisingly, there is a vast literature on this subject (see, e.g., [1] and the referenced bibliography). A completely different direction was taken by Hong and Scaglione [2], who suggested the use of mutually coupled oscillators as the basic mechanism to reach network consensus without the need for sending the data to a fusion center. The principle ensuring the self-synchronization capability of the system proposed in [2], [3] relied on a theorem proved by Mirollo and Strogatz in [4], where the network was supposed to be fully connected. This rather restrictive assumption was later removed by Lucarelli and Wang in [5], who proved that the only really needed property is global connectivity, that is the property that there is a path between each pair of nodes. This was a significant step forward, as it relaxes the need for global coupling, as local coupling is sufficient, provided that the global connectivity is guaranteed. The oscillator and coupling model proposed in [2], [3], and [5] associates the local estimate to the time shift of a pulse oscillator. However, especially for large scale network, this may create a problem, as the information bearing time shift may become indistinguishable from the propagation delay. A more general approach was then proposed in [6] where it was showed how to reach decentralized maximum likelihood estimates through only local coupling, with a scheme which is much more flexible than the ones suggested in [2] or [5]. The aim of this work is to derive the conditions under which the system proposed in [6], [7] is globally asymptotically stable.

This work has been partially funded by ARL, Contract N62558-05-P-0458, and by the Italian Ministry of Education.

2. SELF-SYNCHRONIZATION OF LOCALLY COUPLED OSCILLATORS

The proposed sensor network is composed of N nodes and each node is equipped with four basic components: i) a *transducer* that senses the physical parameter of interest (e.g., temperature, concentration of contaminants, radiation, etc.); ii) a *local detector* or *estimator* that, based on the sensed quantities, takes an initial decision; iii) a *dynamical system* (termed oscillator, for simplicity) whose state evolves in time according to a differential equation which is periodically initialized with the local decision and it is coupled with the states of nearby sensors; iv) a *radio interface* that transmits the state of the associated dynamical systems and receives the state of nearby nodes.

Denoting by ω_i the initial local decision (either the result of a detection or estimation) taken by node i, the dynamical system (oscillator) present in node i evolves according to the following equation

$$\dot{\vartheta}_i(t) = \omega_i + \frac{K}{c_i}\sum_{j=1}^{N} a_{ij}\, f\left(\vartheta_j(t) - \vartheta_i(t)\right),\ i = 1, \ldots, N, \quad (1)$$

where $\vartheta_i(t)$ is the state function of the i-th sensor, that is initialized as a random number $\vartheta_i(0)$; $f(\cdot)$ is, typically, a monotonically increasing nonlinear odd function of its argument that takes into account the mutual coupling between the sensors. Without loss of generality, $f(x)$ is normalized so that $df(0)/dx = 1$. A different value of $df(0)/dx$ can always be included in K; K is a control loop gain; c_i is a coefficient that quantifies the attitude of the i-th sensor to adapt its values as a function of the signals received from the other nodes: The higher is c_i, the less is the attitude of the i-th node to change its original decision ω_i. The running decision, or estimate, of each sensor is encoded in its pulsation $\dot{\vartheta}_i(t)$. The coefficients a_{ij} take into account the local coupling between oscillators. We assume that two oscillators are coupled (i.e., $a_{ij} \neq 0$), only if their distance is smaller than the coverage radius of each sensor[1].

To make explicit the network connectivity properties, it is better to rewrite (1) introducing the so called *incidence* matrix $\boldsymbol{B}$, defined as follows. Given an oriented graph $\mathscr{G}$[2] composed by N vertices and E edges, $\boldsymbol{B}$ is the $N \times E$ matrix such that $[\boldsymbol{B}]_{ij} = 1$ if the edge j is incoming to vertex i, $[\boldsymbol{B}]_{ij} = -1$ if the edge j is outcoming from vertex i, and 0 otherwise. Given the $N \times 1$ vector $\mathbf{1}_N$, composed of all ones, it is easy to check that the incidence matrix satisfies the following property:

$$\mathbf{1}^T \boldsymbol{B} = 0^T. \quad (2)$$

[1]The coverage radius is assumed to be the same for all sensors, even though this could be changed to accommodate for different network topological models, like small worlds or scale-free networks.

[2]An orientation of a graph $\mathscr{G}$ is the assignment of a direction to each edge.

Given $\boldsymbol{B}$, the symmetric $N \times N$ matrix $\boldsymbol{L}$ defined as $\boldsymbol{L} \triangleq \boldsymbol{B}\boldsymbol{B}^T$, is called the *Laplacian* of $\mathscr{G}$, and it is independent of the choice of the orientation. If we associate a positive number w_i to each edge and we build the diagonal matrix $\boldsymbol{D}_{\mathbf{w}} \triangleq \text{diag}(\mathbf{w})$, with $\mathbf{w} \triangleq [w_1, \cdots, w_E]^T$, we may introduce the so called *weighted Laplacian*, which is written as $\mathbf{L}_{\mathbf{w}} \triangleq \mathbf{B}\mathbf{D}_{\mathbf{w}}\mathbf{B}^T$. The second smallest eigenvalue $\lambda_2(\mathbf{L})$ (or $\lambda_2(\mathbf{L}_{\mathbf{w}})$), is referred to as the graph *algebraic connectivity*, and it provides a measure of connectivity.

Using the above notation, $\mathbf{L}_{\mathbf{A}} \triangleq \mathbf{B}\mathbf{D}_{\mathbf{A}}\, \mathbf{B}^T$ will denote the weighted Laplacian associated to the graph describing our network (1), including the positive coefficients $\{a_{ij}\}$. Furthermore, $d_{\max} \triangleq \max_i \sum_{j=1}^N a_{ij}$ will denote the maximum degree of the graph.

Using the incidence matrix $\boldsymbol{B}$, we can rewrite (1) in compact form as

$$\dot{\boldsymbol{\vartheta}}(t) = \boldsymbol{\omega} - K\, \boldsymbol{D}_c^{-1}\, \boldsymbol{B}\, \boldsymbol{D}_A\, f\left(\boldsymbol{B}^T \boldsymbol{\vartheta}(t)\right), \tag{3}$$

where $\boldsymbol{\vartheta}(t) \triangleq [\vartheta_1, \cdots, \vartheta_N]^T$, $\boldsymbol{D}_c \triangleq \text{diag}\{c_1, \ldots, c_N\}$; $\boldsymbol{D}_A$ is an $E \times E$ diagonal matrix, whose diagonal entries are all the weights a_{ij}, indexed from 1 to E; the symbol $f(\boldsymbol{x})$ has to be intended as the vector whose k-th component is $f(x_k)$.

Given (1) (or, equivalently, (3)), we are interested in following solutions.

Definition 1 *The overall population of oscillators (1) is said to synchronize if there exists a vector $\boldsymbol{\vartheta}^\star(t)$, called the synchronized state of the system, such that*

$$\lim_{t \mapsto \infty} \|\dot{\boldsymbol{\vartheta}}_i(t) - \dot{\boldsymbol{\vartheta}}^\star(t)\| = 0, \quad \forall i = 1, 2, \ldots, N, \tag{4}$$

where $\|\cdot\|$ denotes some vector norm. This state is said globally asymptotically stable *if the system synchronizes, for* any *set of initial conditions.*

From Definition 1, it follows that, if there exists a synchronized state that is globally asymptotically stable, then it must necessarily be *unique*. Interestingly, the synchronized state, if it exists, can be computed in closed form, without solving explicitly the system of differential equations. In fact, multiplying (3) by the row vector $\mathbf{c}^T \triangleq \mathbf{1}_N^T \mathbf{D}_c$ from the left side, we obtain

$$\mathbf{c}^T \dot{\boldsymbol{\vartheta}}(t) = \mathbf{c}^T \boldsymbol{\omega} - K\, \mathbf{1}_N^T \mathbf{B}\mathbf{D}_{\mathbf{A}} f\left(\mathbf{B}^T \boldsymbol{\vartheta}\right) = \mathbf{c}^T \boldsymbol{\omega}, \tag{5}$$

where in the second equality of (5), we have used (2). Hence, if system (3) synchronizes (according to Definition 1), the common value of $\dot{\vartheta}^*(t)$ must be equal to

$$\dot{\vartheta}^*(t) \triangleq \omega^* = \frac{\mathbf{c}^T \boldsymbol{\omega}}{\mathbf{1}_N^T \mathbf{c}} = \frac{\sum_{i=1}^N c_i \omega_i}{\sum_{i=1}^N c_i}. \tag{6}$$

3. REACHING GLOBAL ML ESTIMATE THROUGH SELF-SYNCHRONIZATION

The self-synchronization process may form the basic mechanism to reach a global consensus among the network nodes *without a fusion center*. In particular, as shown in [6], self-synchronization may be made to converge to the global optimal maximum likelihood (ML) estimate. We now recast the formulation of [6] in order to emphasize the role of the network incidence matrix. Let us consider the linear observation model, where the i-th sensor observes a vector

$$\boldsymbol{y}_i = \boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{w}_i, \tag{7}$$

where $\boldsymbol{x}$ is the unknown parameter vector, assumed to be the same for all sensors; $\boldsymbol{A}_i$ is the mixing matrix of sensor i, and $\boldsymbol{w}_i$ is the observation noise vector, modeled as a circularly symmetric complex Gaussian vector with zero mean and covariance matrix $\boldsymbol{C}_i$. We assume that the noise vectors affecting different sensors are statistically independent of each other (however, the noise vector present in each sensor may be colored). Let us denote with L the number of unknowns, so that $\boldsymbol{x}$ is a column vector of size L. The observation vector $\boldsymbol{y}_i$ has dimension M. We consider the case where the single sensor must be able, in principle, to recover the parameter vector from its own observation. This requires that $M \geq L$ and that $\boldsymbol{A}_i$ is full column rank. The ML estimate of each sensor alone is then

$$\hat{\boldsymbol{x}}_{ML}^{(i)} = (\boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{A}_i)^{-1} \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{y}_i. \tag{8}$$

An ideal centralized node that gathers all the observation vectors $\boldsymbol{y}_i$ without errors and knows all mixing matrices $\boldsymbol{A}_i$, would derive the optimal centralized ML estimate, equal to

$$\hat{\boldsymbol{x}}_{ML} = \left(\sum_{i=1}^n \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{A}_i\right)^{-1} \left(\sum_{i=1}^n \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{y}_i\right), \tag{9}$$

where the summation extends over all the nodes that send their information to the decision node. Clearly, the estimate (9) is the desired solution, but it is difficult to obtain because it requires a lot of information arriving at the decision node, without errors. In fact, the sensor fusion center would need to know not only all the observations $\boldsymbol{y}_i$, but also the mixing matrices $\boldsymbol{A}_i$ and the noise covariance matrices $\boldsymbol{C}_i$ of each sensor.

We show now that the optimal ML estimate can be achieved through the self synchronization process described before, without the need for collecting all the information in any node. Generalizing the strategy described in the previous section to the vector case, we design nodes that evolve according to the following vector state equation

$$\dot{\boldsymbol{\vartheta}}_i(t) = \hat{\boldsymbol{x}}_{ML}^{(i)} + K(\boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{A}_i)^{-1} \sum_{j=1}^N a_{ij}\, f\left(\boldsymbol{\vartheta}_j(t) - \boldsymbol{\vartheta}_i(t)\right), \tag{10}$$

with $i = 1, \ldots, N$ and $\hat{\boldsymbol{x}}_{ML}^{(i)}$ given by (8). Introducing the vectors $\dot{\boldsymbol{\vartheta}}(t) \triangleq (\dot{\boldsymbol{\vartheta}}_1^T(t), \ldots \dot{\boldsymbol{\vartheta}}_N^T(t))^T$ and $\hat{\boldsymbol{x}} \triangleq (\hat{\boldsymbol{x}}_{ML}^{(1)T}, \ldots, \hat{\boldsymbol{x}}_{ML}^{(N)T})^T$, and the matrices $\boldsymbol{Q}_i \triangleq \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{A}_i$ and $\mathbf{D}_{\mathbf{Q}} \triangleq \text{diag}(\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_N)$, we can rewrite all equations in (10) in a more compact form as

$$\dot{\boldsymbol{\vartheta}}(t) = \hat{\boldsymbol{x}} - K\, \mathbf{D}_{\mathbf{Q}}^{-1}\, \mathbf{P}^T (\boldsymbol{I}_L \otimes \boldsymbol{B}\boldsymbol{D}_{\mathbf{A}})\, f\left[(\boldsymbol{I}_L \otimes \boldsymbol{B}^T)\, \mathbf{P}\, \boldsymbol{\vartheta}(t)\right], \tag{11}$$

where $\mathbf{P}$ is an $LN \times LN$ permutation matrix, such that $[\mathbf{P}]_{ij} = 1$ if $j = ((i-1)L + 1) \bmod N$, and $[\mathbf{P}]_{ij} = 0$ otherwise.

Left-multiplying both sides of (11) by $\left(\mathbf{1}_N^T \otimes \mathbf{I}_L\right) \mathbf{D}_Q$, we obtain

$$\sum_{i=1}^N \boldsymbol{Q}_i\, \dot{\boldsymbol{\vartheta}}_i(t) = \sum_{i=1}^N \boldsymbol{Q}_i \hat{\boldsymbol{x}}_{ML}^{(i)}, \tag{12}$$

where we used the following chain of equalities $(\mathbf{1}_N^T \otimes \mathbf{I}_L)\mathbf{P}^T(\mathbf{I}_L \otimes \mathbf{B}\mathbf{D}_{\mathbf{A}}) = (\mathbf{I}_L \otimes \mathbf{1}_N^T)(\mathbf{I}_L \otimes \mathbf{B}\mathbf{D}_{\mathbf{A}}) = \mathbf{I}_L \otimes \mathbf{1}_N^T \mathbf{B}\mathbf{D}_{\mathbf{A}} = \mathbf{0}$, and the property (2). Hence, if the system has the capability to reach a synchronization state, where $\dot{\boldsymbol{\vartheta}}_i(t) = \dot{\boldsymbol{\vartheta}}^*(t)$, for all i, that it must necessarily be

$$\dot{\boldsymbol{\vartheta}}^*(t) = \left(\sum_{i=1}^n \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{A}_i\right)^{-1} \left(\sum_{i=1}^n \boldsymbol{A}_i^H \boldsymbol{C}_i^{-1} \boldsymbol{y}_i\right). \tag{13}$$

This equilibrium coincides with the global optimal ML estimate (9).

4. GLOBAL ASYMPTOTIC STABILITY OF THE SYNCHRONIZED STATE

Given the dynamic system (1) or (10), some natural questions arise: i) Does the synchronized state exist? ii) If it exists, does the system synchronize, for any set of initial conditions?

In this paper we give an answer to these questions for the system (1). Similar results are obtained for the more general model (10) in [8]. Specifically, we have the following.

Theorem 1 *Given the system (1), assume that the following conditions are satisfied:*

a1 *The graph associated to the network is connected;*

a2 *The nonlinear function $f(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is a continuously differentiable, odd, increasing function in $\mathbb{R}$;*[3].

a3 *The nonzero coefficients a_{ij} are positive.*

Then, there exist two unique critical values of K, denoted by K_L and K_U, with $0 \le K_L \le K_U$, such that the synchronized state exists for all $K > K_U$, *and it does not for* all $K < K_L$. *Furthermore, if it exists, the synchronized state is globally asymptotically stable. Upper and lower bounds of K_L and K_U are*

$$K_L \ge \frac{\|\boldsymbol{D}_{\mathbf{c}}\Delta\boldsymbol{\omega}\|_\infty}{f_{\max} d_{\max}}, \quad \text{and} \quad K_U \le \frac{2\,\|\boldsymbol{D}_{\mathbf{c}}\Delta\boldsymbol{\omega}\|_2}{f_{\max}\lambda_2(\mathbf{L_A})}. \tag{14}$$

where $\Delta\boldsymbol{\omega} \triangleq \boldsymbol{\omega} - \omega^* \mathbf{1_N}$, with ω^* defined in (6); $f_{\max} \triangleq \lim_{x\to+\infty} f(x)$; $d_{\max}$ and $\lambda_2(\mathbf{L_A})$ are the maximum degree and the algebraic connectivity of the graph, respectively.

Proof. See Appendix. ■

Remark 1 Even though conditions in (14) provide only a range for the values of K_L and K_U, they state an important property of the whole system: If we want the network to reach a global consensus (common estimate), it is sufficient to take K greater than the upper bound in (14); conversely, if we do not want the network to reach a global consensus, we need to take K smaller than the lower bound in (14). This was used, for example, in [7] to get spatial smoothing of the observed phenomenon. This is indeed a unique possibility offered by nonlinear systems. Our nonlinear model contains, in fact, as a particular case, linear dynamic systems, corresponding to the choice $f(x) = x$. However, in the case of unbounded coupling function, as in the linear case, the lower and upper bounds in (14) coincide, so that there exists a unique critical value of K, given by $K_L = K_U = 0$. Thus, a linear system always converges to the equilibrium, for any positive values of K. Hence, this is one more example showing that nonlinear systems offer a variety of behaviors impossible for linear systems.

Remark 2 From (14), it is evident that the synchronization properties depend on the graph topology through the second-smallest eigenvalue of the graph representing the network. This means that, for a given K, different topologies give rise to different behaviors. For example, *scale-free* random graphs exhibit an interesting behavior. In fact, denoting with $\bar{\lambda}_2(m_0, N)$ the average value of the second smallest eigenvalue of a network composed of N nodes, averaged over the graph realizations, for a given value of m_0 that is the initial number of nodes used to construct the network according to the iterated procedure of growth and preferential attachment, it was shown in [9] that the limit of $\bar{\lambda}_2(m_0, N)$ for N going to infinity is constant. This proves the *scalability* of scale-free networks, that is the property that, provided that the network size is sufficiently large, adding new nodes does not change the synchronization capabilities and then the possibility to achieve global optimal estimates.

Remark 3 In the particular case of $\boldsymbol{c} = 1$, and under conditions of Theorem, the dynamical system (3) approaches the synchronized state with a speed that is locally proportional to $K\lambda_2(\mathbf{L})$. Once again, this behavior is directly related to the network topology.

[3]For the lack of space, we consider only asymptotically convex or concave functions $f(\cdot)$, i.e. functions that can not change their concavity infinitely often. Observe that this constraint does not represent a strong restriction in the choice of the function $f(\cdot)$. However, the general case is studied in [8]

5. RESULTS AND CONCLUSION

The propagation delays clearly have an impact on the system synchronization. Incorporating the delays explicitly in the system gives rise to the following set of equations

$$\dot{\vartheta}_i(t) = \omega_i + \frac{K}{c_i}\sum_{j=1}^{N} a_{ij}\, f\left(\vartheta_j(t-\tau_{ij}) - \vartheta_i(t)\right), \; i = 1, \ldots, N, \tag{15}$$

where $\tau_{ij} = d_{ij}/c$ is the delay with which the state $\vartheta_j(t)$ of oscillator j reaches oscillator i, d_{ij} is the distance between nodes i and j, and c is the speed of light. This case is not covered by the theory described in this paper. Nevertheless, we have observed by simulation that even with considerable delays, the network converges. However, the final value differs from the one predicted in the absence of delays. The evaluation of this bias is an interesting research topic that we are currently investigating. As a final performance assessment, in Fig. 1, we report the variances obtained in the estimate of a vector parameter of dimension $L = 3$, as a function of the number of sensors N. The observation matrices $\boldsymbol{A}_i$ are 6×3 and their entries are generated as i.i.d. Gaussian random variables, to test the robustness of the proposed approach. The network considered in this case is a regular network, where all nodes have degree 4, for all values of N. It is interesting to observe that, even though the coupling is only local and it does not vary with N, the variance decays as $1/N$, as the optimal ML estimators thus confirming the scalability of the proposed approach.

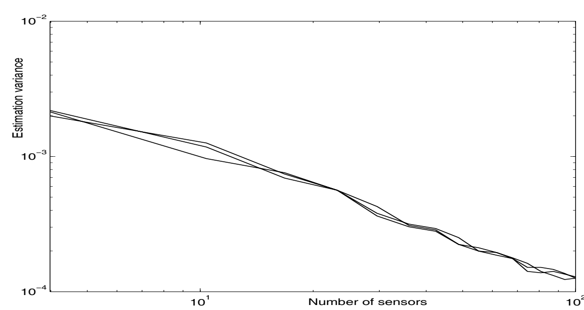

Fig. 1. Estimation variance as a function of the number of sensors.

6. APPENDIX

A complete proof of Theorem 1 is given in [8]. Because of the space limitation, here we provide only the stretch of the proof.

The basic ideas are the following. We introduce, first, a proper transformation of the original system (3), so that the existence and the *global asymptotic* stability (according to Definition 1) of the synchronized state can be recasted in the classical study of existence and the *asymptotic* stability of the equilibria of the transformed system (see, e.g., [10]). Then, we prove, using standard fixed point arguments, that, under $a1$-$a3$, an equilibrium for the transformed system exists, provided that $K > K_U$, and cannot exist if $K < K_L$. Finally, we show, introducing a valid Lyapunov function, that, if an equilibrium exists, it is also asymptotically stable. We need the following intermediate results.

Lemma 2 ([8]) *Given an oriented weighted graph $\mathscr{G}$ with N nodes, and positive numbers $\{w_i\}_i$ associated to the edges, let $\mathbf{L_w} \triangleq \mathbf{B}\mathbf{D_w} \mathbf{B}^T$ be the (weighted) Laplacian of $\mathscr{G}$, where $\mathbf{B}$ is the $N \times E$ incidence matrix, $\mathbf{D_w} \triangleq \operatorname{diag}(\mathbf{w})$ is the $E \times E$ diagonal matrix whose diagonal entries are the edge-weights w_i. Let $\mathbf{L}_\mathbf{w}^\sharp$ denote the generalized inverse of $\mathbf{L_w}$ [11]. If the graph $\mathscr{G}$ is connected, then $\mathbf{L}_\mathbf{w}^\sharp : \mathbb{R}_{++}^E \mapsto \mathbb{R}^N$ is a continuous function in $\mathbb{R}_{++}^E$.*

Lemma 3 ([12, Theorem 4.14]) *Let $\mathcal{C}$ be a closed, convex subset of a normed linear space. Then, every compact[4], continuous map $F : \mathcal{C} \mapsto \mathcal{C}$ admits at least one fixed point.*

Existence. Assume that conditions $a1$ and $a3$ are satisfied and consider the following change of variables $\Psi_i(t) = \vartheta_i(t) - \omega^* t$, with ω^* defined in (6). The original system (3) can be equivalently rewritten as

$$\begin{aligned}\dot{\boldsymbol{\Psi}}(t) &= \Delta\boldsymbol{\omega} - K\,\mathbf{D}_\mathbf{c}^{-1}\mathbf{B}\mathbf{D}_A f\left[\mathbf{B}^T\boldsymbol{\Psi}(t)\right] \\ &= \Delta\boldsymbol{\omega} - K\mathbf{D}_\mathbf{c}^{-1}\mathbf{B}\mathbf{D}_A\mathbf{D}_\boldsymbol{\Psi}\mathbf{B}^T\boldsymbol{\Psi}(t) \\ &\triangleq \Delta\boldsymbol{\omega} - K\mathbf{D}_\mathbf{c}^{-1}\mathbf{L}_\boldsymbol{\Psi}\boldsymbol{\Psi}(t) \qquad (16)\end{aligned}$$

where $\boldsymbol{\Psi}(t) = [\Psi_1(t), \ldots, \Psi_N(t)]^T$, with $\boldsymbol{\Psi}(0) = \boldsymbol{\vartheta}(0) \in [-a, a]^N$, $\Delta\boldsymbol{\omega} = \boldsymbol{\omega} - \omega^*\mathbf{1}$, and $\mathbf{L}_\boldsymbol{\Psi} \triangleq \mathbf{B}\mathbf{D}_A\mathbf{D}_\boldsymbol{\Psi}\mathbf{B}^T$ is the weighted Laplacian of the graph, with diagonal weights-matrix $\mathbf{D}_A\mathbf{D}_\boldsymbol{\Psi}$ (that depend on $\boldsymbol{\Psi}(t)$), and $[\mathbf{D}_\boldsymbol{\Psi}]_{ii}$ given by

$$[\mathbf{D}_\boldsymbol{\Psi}]_{ii} = \frac{f\left([\mathbf{B}^T\boldsymbol{\Psi}]_i\right)}{[\mathbf{B}^T\boldsymbol{\Psi}]_i} > 0, \quad i = 1, \ldots, E, \qquad (17)$$

where the positivity of $[\mathbf{D}_\boldsymbol{\Psi}]_{ii} > 0$ for all $\boldsymbol{\Psi}$, comes from $a2$.

The synchronized state for (3) exists if and only if (16) admits an equilibrium, or equivalently, the following fixed point equation admits a solution

$$\mathbf{L}_\Psi\boldsymbol{\Psi} = \frac{\mathbf{D}_c\Delta\boldsymbol{\omega}}{K}. \qquad (18)$$

For bounded functions $f(\cdot)$, i.e. $f^{\max} < \infty$, the left side of (18) is bounded. Hence, there exists a sufficiently low K such that $K\left\|\mathbf{L}_\Psi\ \ \boldsymbol{\Psi}\right\|_\infty < \left\|\mathbf{D}_c\Delta\boldsymbol{\omega}\right\|_\infty$, which guarantees the existence of K_L, and the lower bound in (14)[5]. In the case of unbounded $f(\cdot)$ (as, e.g., for linear dynamic systems), the lower bound of K_L disappears.

We prove the existence of K_U, computing directly the upper bound in (14). To this end, we use Lemma 3, and show that the equation (18) admits a fixed-point in some compact, convex set of $\mathbb{R}^N$, chosen, without loss of generality, as $\mathcal{B}_a \triangleq \left\{\mathbf{x} \in \mathbb{R}^N : \|\mathbf{x}\|_2 \leq a\right\}$, where a is any number in $\mathbb{R}_{++}$. Invoking Lemma 2 and denoting by $\mathbf{L}_\boldsymbol{\Psi}^\sharp$ the generalized (Drazin) inverse of the weighted Laplacian $\mathbf{L}_\boldsymbol{\Psi}$ [11], we have that the mapping $\mathbf{L}_{\mathbf{A},\boldsymbol{\Psi}}^\sharp\mathbf{D}_\mathbf{c}\Delta\omega/K$ in (18) is continuous on $\mathbb{R}^N$ (because of the positivity of $[\mathbf{D}_\boldsymbol{\Psi}]_{ii} > 0$, $\forall\boldsymbol{\Psi} \in \mathbb{R}^N$). Hence, according to Lemma 3, a fixed-point for (18) exists, if $\mathbf{L}_{\mathbf{A},\boldsymbol{\Psi}}^\sharp\mathbf{D}_\mathbf{c}\Delta\omega/K$ is a compact map on $\mathcal{B}_a$, for some $a \in \mathbb{R}_{++}$. This is guaranteed if, for any given $a \in \mathbb{R}_{++}$, K in (18) is chosen so that $\|\mathbf{L}_{\mathbf{A},\boldsymbol{\Psi}}^\sharp\mathbf{D}_\mathbf{c}\Delta\boldsymbol{\omega}\|_2 \leq K\,a$, which corresponds to

$$K \geq \frac{\|\mathbf{L}_\boldsymbol{\Psi}^\sharp\|_2\,\|\mathbf{D}_c\Delta\boldsymbol{\omega}\|_2}{a} = \frac{\|\mathbf{D}_c\Delta\boldsymbol{\omega}\|_2}{a\,\lambda_2\left(\mathbf{L}_\boldsymbol{\Psi}\right)}, \qquad (19)$$

where $\|\mathbf{L}_\boldsymbol{\Psi}^\sharp\|_2$ is the spectral norm of $\mathbf{L}_\boldsymbol{\Psi}^\sharp$, and $\lambda_2\left(\mathbf{L}_\boldsymbol{\Psi}\right)$ is algebraic connectivity of $\mathbf{L}_\boldsymbol{\Psi}$. In order to remove the dependence of $\lambda_2\left(\mathbf{L}_\boldsymbol{\Psi}\right)$ on $\boldsymbol{\Psi}$, we consider the more stringent condition

$$K \geq \frac{\|\mathbf{D}_c\Delta\boldsymbol{\omega}\|_2}{\min_{\boldsymbol{\Psi}\in\mathcal{B}_a}\left\{a\,\lambda_2\left(\mathbf{L}_\boldsymbol{\Psi}\right)\right\}}, \qquad (20)$$

Using $\min_{\boldsymbol{\Psi}\in\mathcal{B}_a}\left\{a\,\lambda_2\left(\mathbf{L}_\boldsymbol{\Psi}\right)\right\} = \lambda_2\left(\mathbf{L}_A\right)\min_{\mathbf{x}\in[0,2a]}\left\{a\,f(x)/x\right\}$ and $\sup\left\{a\min_{\mathbf{x}\in[0,2a]}\ \ f(x)/x\right\} = f_{\max}/2$, we obtain the upper bound of K_U in (14) [8].

Global stability of the synchronized state. Assume now that $K > K_U$ so that system (3) may synchronize. After rewriting (3) as in (16), it is straightforward to check that the synchronized state for (3) is globally asymptotically stable (according to Definition 1) if and only if system (16) converges to an equilibrium, for any set of initial conditions. We showed in [8] that this occurs if the point $\boldsymbol{\Psi} = 0$ is the globally asymptotically stable equilibrium of the following related system:

$$\dot{\boldsymbol{\Psi}}(t) = -K\mathbf{D}_\mathbf{c}^{-1}\bar{\mathbf{L}}_\boldsymbol{\Psi}\boldsymbol{\Psi}(t), \quad \mathbf{1}_N^T\mathbf{D}_c\boldsymbol{\Psi}(t) = 0, \qquad (21)$$

where $\bar{\mathbf{L}}_\boldsymbol{\Psi} \triangleq \mathbf{B}\mathbf{D}_A\bar{\mathbf{D}}_\boldsymbol{\Psi}\mathbf{B}^T$ with $\bar{\mathbf{D}}_\boldsymbol{\Psi}$ diagonal matrix, whose diagonal entries are the (positive) weights $g_{ij}(\Psi_j - \Psi_i)/(\Psi_j - \Psi_i)$ indexed from 1 to E, and $g_{ij}(\cdot)$ is a function related to $f(\cdot)$ [8], with the following properties, $\forall i \neq j$: i) $g_{ij}(x) = -g_{ji}(-x)$; ii) $xg_{ij}(x) > 0, \forall x \neq 0$ and $xg_{ij}(x) = 0, \Leftrightarrow x = 0$.

Using properties i) and ii) of $g(\cdot)$, it can be shown that the function $V(\boldsymbol{\Psi}) = \|\ \mathbf{D}_\mathbf{c}\boldsymbol{\Psi}\ \|^2$ is a positive definite Lyapunov function [8] for (21), which proves the globally asymptotic stability of the equilibrium $\boldsymbol{\Psi} = \mathbf{0}$ of (21).

[4]The map $f : \mathcal{C} \mapsto \mathcal{C}$ is called compact if $f(\mathcal{C})$ is contained in a compact subset of $\mathcal{C}$.

[5]Observe that the matrix norm induced by the vector infinity norm is the maximum among the absolute values of the row sums [11], i.e. $d^{\max}$.

7. REFERENCES

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