MMSE EQUALIZATION FOR ZERO PADDED MULTICARRIER SYSTEMS WITH INSUFFICIENT GUARD LENGTH

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ABSTRACT

We derive a block linear MMSE equalizer suitable for zero padded multicarrier systems with insufficient guard length. The proposed equalizer is not limited to systems using the IFFT/FFT pair (like in DMT or OFDM), but to any multicarrier system that can be modelled as a pair of perfect reconstruction filter banks. As the computational complexity of the equalizer can be high, an efficient implementation based on the Cholesky factorization is developed. Compared to the traditional zero-forcing approach, experiments show that in the presence of interblock interference (IBI), the proposed equalizer performs significantly better, regardless of the signal-to-noise ratio (SNR). In the absence of IBI (i.e. when the guard interval is sufficiently long), improvements have also been observed at low SNR.

1. INTRODUCTION

In recent years, multicarrier systems have been successfully used in consumer products such as ADSL (asymmetric digital subscriber line) modems and wireless broadband routers. Multicarrier modulation subdivides the channel bandwidth into several narrow band subcarriers; portions of the input bit stream are then allocated to each subcarrier and transmitted independently. The equalization scheme used in these systems usually relies on guard intervals, either in the form of zero padding or a cyclic prefix. To operate properly, the duration of the guard interval must be longer than the channel impulse response. This may pose problems in applications where the channel impulse response is long because a notable fraction of the available bandwidth will have to be reserved for equalization purposes. For instance, this problem arises with the deployment of DSL services in rural areas, as the copper telephone loops can be much longer than those in urban agglomerations.

There is thus a need to develop an equalization scheme that does not necessarily impose a minimum guard length. This problem has been investigated in [1] from a zero-forcing point of view, i.e. where the received symbols are forced to be equal to transmitted symbols, regardless of noise. Yet the equalizer proposed in [1] is not a "true" zero-forcing equalizer due to certain approximations proposed by the authors. In this work, we consider the minimum mean square error (MMSE) approach, where the goal is to minimize the error power between the received and transmitted symbols. Such an approach has been studied extensively in [2], but only for systems having guard intervals of sufficient length.

We derive a MMSE equalizer suitable for zero padded multicarrier systems with insufficient guard length. The channel impulse response is assumed to be known. The proposed equalizer is not limited to systems using the IFFT/FFT pair (like in DMT or OFDM), but to any multicarrier system that can be modelled as a pair of perfect reconstruction filter banks. As the computational complexity of the equalizer can be high, an efficient implementation based on the Cholesky factorization is discussed. Compared to the zero-forcing approach, simulations indicate that the proposed solution can improve the achievable bit rates significantly, especially when the guard length is insufficient.

We use the following notation. Matrices $\mathbf{0}_{M\times N}$ and $\mathbf{I}_M$ denote the $M\times N$ zero matrix and the $M\times M$ identity matrix, respectively. We use $\mathbf{A}[i,j]$ to refer to the (i,j) entry of matrix $\mathbf{A}$, while $\mathbf{A}[i,:]$ and $\mathbf{A}[:,j]$ respectively denote the i-th row and the j-th column of $\mathbf{A}$.

2. ZERO PADDED MULTICARRIER SYSTEMS

The multicarrier transceiver system considered in this paper is illustrated in Fig. 1. The block of M transmitted symbols are represented by the vector $\mathbf{x}[n]$,

$$\mathbf{x}[n] \triangleq \begin{pmatrix} x_0[n] & x_1[n] & \dots & x_{M-1}[n]\end{pmatrix}^T,$$

whose elements are symbols typically chosen from a QAM constellation according to some bit loading rule. Modulation is performed via the $K\times M$, $K\geq M$, polynomial matrix $\mathbf{G}(z)$, representing a filter bank. A block of J zeros is then appended to the modulated time-domain samples to combat interblock interference (IBI), as detailed later. This zero padded sequence is sent through a channel modelled by a finite impulse response (FIR) filter $C(z)$ of degree Q, i.e.

$$C(z) = \sum_{n=0}^{Q} c[n]z^{-n}.$$

An additive noise $\eta[m]$ is also present in the channel output. At the receiver end, equalization is carried out via a block linear equalizer represented by the $K\times N$ matrix $\mathbf{E}$, where $N \triangleq K+J$. Finally, the $M\times K$ filter bank $\mathbf{S}(z)$ demodulates the equalized data, yielding the received block

$$\hat{\mathbf{x}}[n] \triangleq \begin{pmatrix} \hat{x}_0[n] & \hat{x}_1[n] & \dots & \hat{x}_{M-1}[n]\end{pmatrix}^T.$$

To ease mathematical treatment, we also define the $N\times N$ polynomial channel matrix

$$\mathbf{C}(z) = \sum_{n=0}^{Q'} \mathbf{C}_n z^{-n}$$

which results from cascading the decimators, delay elements, $C(z)$, expanders and advance elements (see Fig. 1). $\mathbf{C}(z)$ can be obtained

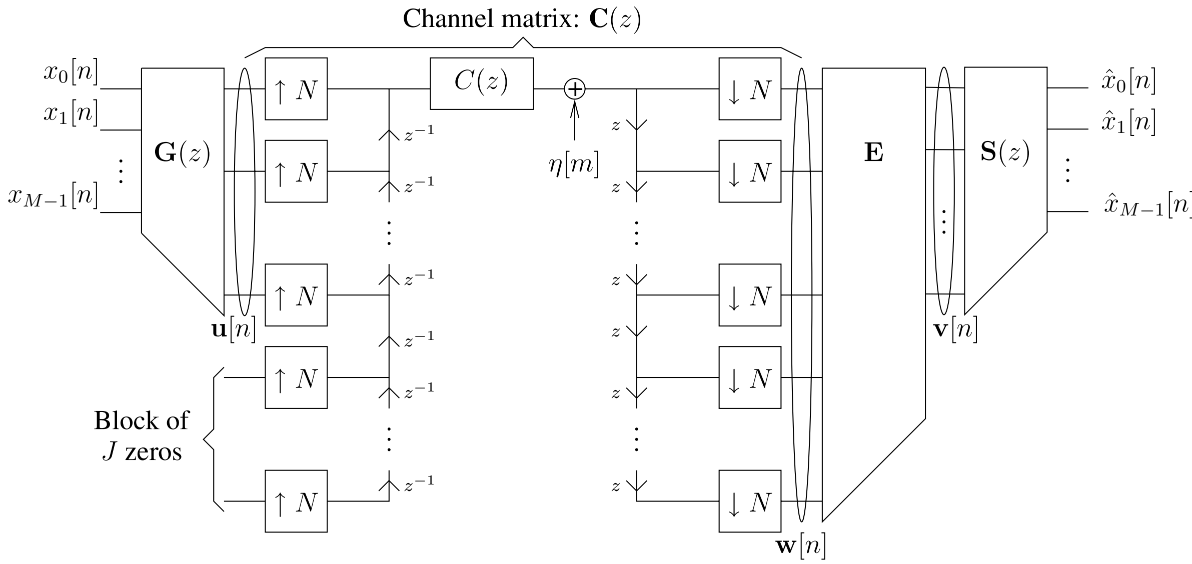

Fig. 1. A zero padded filter bank transceiver.

by using the so-called polyphase identity [3]. In addition, we use $K \times 1$ vectors $\mathbf{u}[n]$ and $\mathbf{v}[n]$ to denote the modulator output and the demodulator input, respectively. Hence, we can write

$$\mathbf{v}[n] = \mathbf{E}\left(\sum_{k=0}^{Q'} \mathbf{C}_k \begin{pmatrix} \mathbf{u}[n-k] \\ \mathbf{0}_{J\times 1} \end{pmatrix} + \boldsymbol{\eta}[n]\right), \tag{1}$$

where

$$\boldsymbol{\eta}[n] \triangleq \begin{pmatrix} \eta[Nn] & \eta[Nn+1] & \dots & \eta[Nn+N-1] \end{pmatrix}^T.$$

Throughout this paper, the following assumptions are made:

1. The filter banks $\mathbf{G}(z)$ and $\mathbf{S}(z)$ are already designed such that $\mathbf{S}(z)\mathbf{G}(z) = z^{-d}\mathbf{I}_M$ (where d is some delay, assumed to be zero without loss of generality), i.e. they are characterized by the perfect reconstruction property.
2. We have $Q < N$, $N = K + J$.
3. The channel $C(z)$ is known.

The first assumption allows us to re-use one of the several available methods of designing perfect reconstruction filter banks [3]. As an example, DFT filter banks could be employed [4] or one could simply use the DFT matrix as in OFDM / DMT [5]. Although this approach is suboptimal from a MMSE point of view, where the goal is to minimize the average power of $\hat{\mathbf{x}}[n] - \mathbf{x}[n]$, it simplifies the design problem greatly since the filter banks and the equalizer can be considered as two independent design problems. Nevertheless, as shown in Sec. 5, the proposed suboptimal approach improves performance considerably compared to the traditional design.

The second assumption guarantees that the channel matrix $\mathbf{C}(z)$ can be decomposed as follows [2]

$$\mathbf{C}(z) = \mathbf{C}_0 + \mathbf{C}_1 z^{-1}, \tag{2}$$

where $\mathbf{C}_0$ is a lower triangular Toeplitz matrix whose first column is given by

$$\mathbf{C}_0[:,1] = \begin{pmatrix} c[0] & \dots & c[Q] & 0 & \dots & 0 \end{pmatrix}^T$$

and $\mathbf{C}_1$ is an upper triangular Toeplitz matrix whose first row is given by

$$\mathbf{C}_1[1,:] = \begin{pmatrix} 0 & \dots & 0 & c[Q] & \dots & c[1] \end{pmatrix}.$$

Note that the condition on Q (i.e. $Q < N$) is far less restrictive than what is usually permitted (e.g. see [2]). As shown below, zero-forcing equalizers require that $Q \leq J$ to ensure complete intersymbol interference (ISI) cancellation.

One of the most common approach to equalizer design would be the well-known zero-forcing solution. Such equalizer, denoted here by $\mathbf{E}_{\text{ZF}}$, ignores the presence of noise completely (i.e. $\boldsymbol{\eta}[n] = \mathbf{0}_{N\times 1}$) and enforces $\mathbf{v}[n] = \mathbf{u}[n]$. Hence, from (1) and (2), we have

$$\mathbf{v}[n] = \mathbf{E}_{\text{ZF}}\left(\mathbf{C}_0 \begin{pmatrix} \mathbf{u}[n] \\ \mathbf{0}_{J\times 1} \end{pmatrix} + \mathbf{C}_1 \begin{pmatrix} \mathbf{u}[n-1] \\ \mathbf{0}_{J\times 1} \end{pmatrix}\right). \tag{3}$$

Note that $\mathbf{C}_1$ is responsible for IBI since it allows the previously transmitted block, $\mathbf{u}[n-1]$, to be combined with $\mathbf{u}[n]$ to yield the current received block. Now, if we partition $\mathbf{C}_n$ $(n = 0, 1)$ as

$$\mathbf{C}_n = \begin{pmatrix} \mathbf{C}_{n,0} & \mathbf{C}_{n,1} \end{pmatrix}, \tag{4}$$

where $\mathbf{C}_{n,0}$ is a $N \times K$ matrix, then (3) becomes

$$\begin{aligned} \mathbf{v}[n] &= \mathbf{E}_{\text{ZF}}\left(\mathbf{C}_{0,0}\mathbf{u}[n] + \mathbf{C}_{1,0}\mathbf{u}[n-1]\right) \\ &= \mathbf{E}_{\text{ZF}}\mathbf{C}_{0,0}\mathbf{u}[n] \quad \text{if } Q \leq J. \end{aligned} \tag{5}$$

In this case, a zero-forcing solution can be found by using the pseudo-inverse (denoted by the superscript †), i.e.

$$\mathbf{E}_{\text{ZF}} = \mathbf{C}_{0,0}^{\dagger}. \tag{6}$$

Such solution can be obtained by the QR factorization since $\mathbf{C}_{0,0}$ is full rank (due to its Toeplitz nature). However, when the guard length is insufficient, i.e. if $Q > J$ in (5), a true zero-forcing equalizer cannot be found due to the presence of IBI.

3. MMSE WITH INSUFFICIENT GUARD LENGTH

There are two main reasons to consider a MMSE solution to the equalization problem instead of the zero-forcing approach. The first reason is to allow the guard length J to be less than the channel degree Q. As mentioned in Sec. 2, no zero-forcing solution exists in this case. The second reason is to be able to compensate for noise. Indeed, to ensure reliable digital transmissions, we expect $\mathbf{v}[n]$ to be as close as possible to $\mathbf{u}[n]$. In a zero-forcing approach, noise could be amplified so much that the aforementioned does not hold.

If noise is no longer ignored, we can write, from (1) and (2),

$$\mathbf{v}[n] = \mathbf{E}\left(\mathbf{C}_{0,0}\mathbf{u}[n] + \mathbf{C}_{1,0}\mathbf{u}[n-1] + \boldsymbol{\eta}[n]\right), \tag{7}$$

where $\mathbf{C}_{0,0}$ and $\mathbf{C}_{1,0}$ are the matrix partitions defined in (4). The goal of the MMSE approach is to find a linear transformation $\mathbf{E} = \mathbf{E}_{\text{MMSE}}$ such that the average power of $\mathbf{e}[n]$, given by

$$\mathbf{e}[n] = \mathbf{v}[n] - \mathbf{u}[n], \tag{8}$$

is minimized. This minimization problem can be solved using partial derivatives as follows

$$\frac{\partial\,\mathcal{E}\left(\operatorname{tr}\left(\mathbf{e}[n]\mathbf{e}^T[n]\right)\right)}{\partial\,\mathbf{E}_{\text{MMSE}}} = \mathbf{0}, \tag{9}$$

where $\mathcal{E}$ and tr respectively denote the expectation and trace (i.e. the sum of the diagonal elements) operators.

Using the matrix differentiation rules in [6], if we substitute (7) and (8) into (9), we can solve for $\mathbf{E}_{\text{MMSE}}$ and obtain

$$\mathbf{E}_{\text{MMSE}} = \mathbf{R}_u\mathbf{C}_{0,0}^T\big(\mathbf{C}_{0,0}\mathbf{R}_u\mathbf{C}_{0,0}^T + \mathbf{C}_{1,0}\mathbf{R}_u\mathbf{C}_{1,0}^T + \mathbf{R}_\eta\big)^{-1}, \tag{10}$$

where

$$\mathbf{R}_u \triangleq \mathcal{E}\mathbf{u}[n]\mathbf{u}^T[n]$$

and

$$\mathbf{R}_\eta \triangleq \mathcal{E}\boldsymbol{\eta}[n]\boldsymbol{\eta}^T[n].$$

The MMSE equalizer found in (10) is similar in form to the one derived in [2], except that imperfect IBI mitigation (due to insufficient guard length) is now taken into account via the term $\mathbf{C}_{1,0}\mathbf{R}_u\mathbf{C}_{1,0}^T$.

4. FAST IMPLEMENTATION OF THE MMSE EQUALIZER

The proposed MMSE equalizer given by (10) can be costly (in terms of computational power) to implement, especially in wireless environments where the channel impulse response is time varying. One effective technique to reduce the computational complexity is to make use of the diagonalization property of circulant matrices [5]. However, such technique requires "extending" the Toeplitz matrix $\mathbf{C}_{0,0}$ into a circulant structure and can only be applied in situations where the guard length is sufficiently long (i.e. $J \geq Q$). Moreover, it is necessary to assume that the additive noise is white. A different technique must be considered for systems contaminated by coloured noise or with insufficient guard length (i.e. $J < Q$). For these systems, we derive an efficient implementation based on the Cholesky factorization by exploiting the structure of $\mathbf{C}_{n,0}$ ($n = 0, 1$) and the fact that $\mathbf{R}_\eta$ is symmetric and positive definite.

We will assume that the modulator generates uncorrelated samples, i.e. $\mathbf{R}_u = \sigma_u^2\mathbf{I}_K$. The MMSE equalizer thus becomes

$$\mathbf{E}_{\text{MMSE}} = \sigma_u^2\mathbf{C}_{0,0}^T\mathbf{D}^{-1},$$

where

$$\mathbf{D} = \sigma_u^2\left(\mathbf{C}_{0,0}\mathbf{C}_{0,0}^T + \mathbf{C}_{1,0}\mathbf{C}_{1,0}^T\right) + \mathbf{R}_\eta. \tag{11}$$

We can show that, due to the nature of the matrices involved in (11), $\mathbf{D}$ is symmetric and positive definite. As such, the Cholesky factorization can be employed to "invert" $\mathbf{D}$ [7]. A complete procedure which computes $\mathbf{v}[n]$, the output of the proposed linear block equalizer, i.e.

$$\mathbf{v}[n] = \mathbf{E}_{\text{MMSE}}\mathbf{w}[n],$$

is given by Algorithm 1. Vector $\mathbf{w}[n]$ represents the input of the block equalizer (see Fig. 1) and is defined as

$$\mathbf{w}[n] = \mathbf{C}_{0,0}\mathbf{u}[n] + \mathbf{C}_{1,0}\mathbf{u}[n-1] + \boldsymbol{\eta}[n].$$

The amount of *flops* (floating point operations) for each step of the algorithm is also given.

Algorithm 1 Compute $\mathbf{v}[n] = \mathbf{E}_{\text{MMSE}}\mathbf{w}[n]$.

1: Form $\mathbf{D} = \sigma_u^2(\mathbf{C}_{0,0}\mathbf{C}_{0,0}^T + \mathbf{C}_{1,0}\mathbf{C}_{1,0}^T) + \mathbf{R}_\eta$
   $\{\frac{11}{2}N^2 - 2NK + K^2$ flops$\}$
2: Compute the Cholesky factorization of $\mathbf{D}$, i.e. $\mathbf{D} = \mathbf{U}^T\mathbf{U}$
   $\{\frac{1}{3}N^3$ flops$\}$
3: Solve $\mathbf{w}[n] = \mathbf{D}\mathbf{x}[n] = \mathbf{U}^T\mathbf{U}\mathbf{x}[n]$ using back substitutions
   $\{2N^2$ flops$\}$
4: Evaluate $\mathbf{v}[n] = \mathbf{E}_{\text{MMSE}}\mathbf{w}[n] = \sigma_u^2\mathbf{C}_{0,0}^T\mathbf{x}[n]$
   $\{2NK$ flops (without exploiting structure)$\}$

Notice that the product $\mathbf{C}_{0,0}\mathbf{C}_{0,0}^T$ in (11) involves highly structured matrices (being both Toeplitz and triangular) and can be computed very efficiently using Algorithm 2. A similar algorithm can also be developed to compute $\mathbf{C}_{1,0}\mathbf{C}_{1,0}^T$. If the structure of $\mathbf{C}_{0,0}$ is disregarded, about $2N^2K$ flops would be required for such operation, whereas Algorithm 2 necessitates $\frac{3}{2}N^2 - NK + \frac{1}{2}K^2$ flops (if $N = K$, this amounts to N^2 flops). Note that further optimizations are possible since $\mathbf{C}_{0,0}$ has lower bandwidth Q. The complexity of Algorithm 1 is thus dominated by the Cholesky factorization which requires $\frac{1}{3}N^3$ flops [7]. In comparison, an implementation using the LU factorization would involve $\frac{2}{3}N^3 + 4N^2K$ flops.

Algorithm 2 Compute $\mathbf{B} = \mathbf{C}_{0,0}\mathbf{C}_{0,0}^T$.

```
1: Let D = C_{0,0}(:,1) C_{0,0}^T(:,1)
   {1/2 N^2 flops}
2: B = D
3: for i = 2 to K do
4:   for j = i to N do
5:     B(i,j) = B(j,i) = B(i,j) + B(i-1,j-1)
       {NK - 1/2 K^2 flops}
6:   end for
7: end for
8: for i = K+1 to N do
9:   for j = i to N do
10:    B(i,j) = B(j,i) = B(i,j) + B(i-1,j-1) - D(i-K, j-K)
       {(N-K)^2 flops}
11:  end for
12: end for
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5. EXPERIMENTAL RESULTS

We now assess the performance of the proposed MMSE equalizer. For this purpose, we focus our attention on a multicarrier system

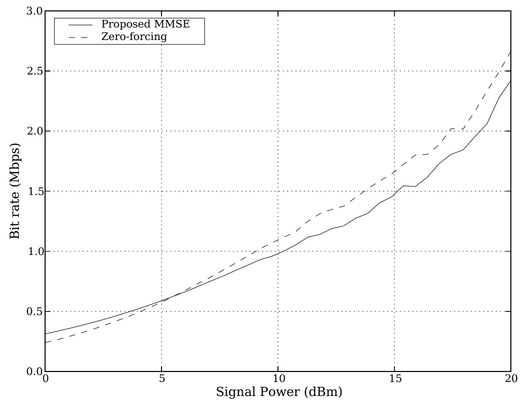

Fig. 2. Average bit rates obtained for $\alpha = 0.2$, $Q = 16$, $J = 16$ (sufficient guard length).

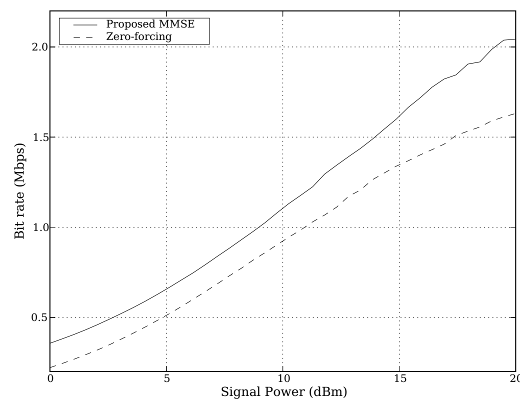

Fig. 3. Average bit rates obtained for $\alpha = 0.2$, $Q = 16$, $J = 4$ (46% of transmitted power is outside the guard interval).

with $M = 64$ and $K = 72$, where $\mathbf{G}(z)$ and $\mathbf{S}(z)$ both represent redundant perfect reconstruction DFT filter banks. The filter banks are characterized by a prototype filter of 1600 taps in total (25 taps per subcarrier) which is designed using the procedure described in [4]. The transceiver is simulated in a DSL-like environment. The sampling rate is set to 2.208 MHz and the channel is time-invariant.

Experiments are conducted using a white Gaussian noise characterized by a flat power spectral density (PSD) of -55 dBm/Hz. Performance is measured in terms of achievable bit rates as obtained using the procedure outlined in [4]. For comparison purposes, both the proposed MMSE and zero-forcing equalizers, respectively given by (10) and (6), are simulated. Moreover, we consider two scenarios:

1. $Q = 16$ and $J = 16$, i.e. the guard interval has sufficient length.
2. $Q = 16$ and $J = 4$, i.e. we use the same channel as in the first case, but we shorten the guard length significantly. Zero padding is then unable to cancel IBI completely.

Results are presented in Figs. 2 and 3, where we show bit rates averaged over 100 randomly generated channels. Such channels are created using an exponentially damped sequence of random variables, i.e.

$$c[n] = e^{-\alpha n} X[n], \quad n = 0, \ldots, Q,$$

where $X[n]$ is a sequence of independent identically distributed Gaussian random variables with zero-mean and unit variance.

By considering the case illustrated in Fig. 2 (sufficient guard length), we notice that the proposed MMSE equalizer performs better than the zero-forcing one at low signal-to-noise ratios (SNR). However, at high SNR, an opposite observation can be made. Such response is a consequence of the fact that our MMSE approach does not minimize the error signal of the overall system, but only from the modulator output to the demodulator input. In this situation, one could design a hybrid equalizer that switches from the MMSE solution to the zero-forcing solution depending on the detected noise level. When the guard length is insufficient, as shown in Fig. 3, the MMSE equalizer significantly outperforms the zero-forcing equalizer regardless of the SNR. For instance, with a 15 dBm signal power, the achievable bit rate can be improved by about 23%. The MMSE equalizer, in this case, is able to compensate for noise at low SNR and to counterbalance IBI at high SNR.

6. CONCLUSION

In this paper, we have developed a block linear equalizer based on the MMSE criterion that is suitable for a zero padded multicarrier system. We did not necessarily assume sufficient guard length and thus IBI was considered during the design process. We have also derived a fast implementation based on the Cholesky factorization. Experiments show that in the presence of IBI, the proposed equalizer performs significantly better than the zero-forcing one. Possible future work includes the consideration of the true MMSE between $\hat{\boldsymbol{x}}[n]$ and $\boldsymbol{x}[n]$.

7. REFERENCES

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