ANALYSIS OF THE PEAK-TO-AVERAGE POWER RATIO OF THE OVERSAMPLED OFDM

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ABSTRACT

The oversampled OFDM modulation is a possible alternative to conventional OFDM for the transmission of signals over multipath fading channels. Indeed with oversampled OFDM, appropriate pulse-shaping can be introduced to fight against time and frequency dispersion. In this paper, we propose a theoretical and experimental analysis of the peak-to-average power ratio (PAPR) of oversampled OFDM. Our analysis illustrates the impact of the oversampling ratio and of the pulse shape on the PAPR distribution.

1. INTRODUCTION

Orthogonal Frequency Division Multiplexing (OFDM) corresponds to a multi-carrier modulation (MCM) scheme that is now widely used for signal transmission over multipath fading channels. Indeed the decomposition of the OFDM signal over several narrow bands in frequency is an appropriate technique to fight against the frequency selectivity that is inherent, for instance, to the mobile radio channel. However, even if OFDM is now part of various transmission standards and also a candidate for future ones [1], still many efforts are necessary to get even more efficient modulation schemes. For instance, conventional OFDM, that can be seen as a Gabor system with a rectangular pulse shape, naturally leads to high out-of-band energy radiations. To get a Gabor system with a better pulse shape, we have to relax the maximum spectral efficiency constraint. This key result in time-frequency analysis is mathematically funded on the Balian-Low theorem [2]. The practical implication of Balian-Low theorem is that either the symbol duration T_0, or the frequency spacing F_0, has to be increased to get a pulse shape being well-localized in the time-frequency plane. Naturally, increasing F_0 reduces the spectral efficiency, but then the advantage compared to OFDM with an increased symbol duration due to the cyclic prefix (CP-OFDM) is the possibility to obtain well-localized pulses. We, then, get oversampled OFDM systems that may be derived, as in [3], from a continuous-time pulse shape, or can be directly designed in discrete-time [4–6].

Therefore, as any MCM, oversampled OFDM produces a non-constant envelope signal, which may be a serious drawback w.r.t. the power amplification. Indeed, signals with large dynamic range may introduce various distortions [7]. Then, the potential loss in power amplification can be measured by the peak-to-average power ratio (PAPR). Until now, there are only a few publications analyzing the impact of a given waveform on the PAPR. Reference [8] concerns the case of pulse shaping for CP-OFDM; in [9], we propose an analysis devoted to another alternative of OFDM known as OFDM with offset quadrature amplitude modulation (OFDM/OQAM), see e.g. [2]. Our aim in this paper is to extend to oversampled OFDM the PAPR analysis provided in [9] for OFDM/OQAM.

In section 2, we give a short description of the discrete-time oversampled OFDM. Then, in section 3, we provide an approximate expression of the complementary cumulative density function (CCDF) for the PAPR. Finally, in section 4, we introduce a measure to predict the CCDF evolution w.r.t. the pulse shape and to the oversampling ratio.

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2. SYSTEM DESCRIPTION

A discrete-time oversampled OFDM modulation with M carriers can be derived from a Gabor system satisfying the Balian-Low requirement, i.e. the density d of time-frequency lattice is less than 1 ($d < \frac{1}{F_0 T_0}$). That means we have N samples per symbol, with $N > M$, leading to an oversampling ratio $\eta = d^{-1} = \frac{N}{M}$. Then, the baseband equivalent of the oversampled OFDM signal can be written as follows [4–6]:

$$s[k] = \sum_{m=0}^{M-1} \sum_{n \in \mathbf{Z}} c_{m,n} h[k-nN] e^{j\frac{2\pi}{M}m(k-D/2)}, \qquad (1)$$

with k the time index, $D = L_h - 1$ with L_h the length of the prototype filter h, i.e. the pulse shaping filter, and $c_{m,n}$ the transmitted complex symbols. The orthogonality of the family of discrete-time functions:

$$h_{m,n}[k] = h[k-nN] e^{j\frac{2\pi}{M}m(k-D/2)} \qquad (2)$$

can be checked with the inner product:

$$\langle f, g \rangle = \sum_{k \in \mathbf{Z}} f[k] g^*[k]. \qquad (3)$$

3. THE PAPR FOR THE OVERSAMPLED OFDM

The signal $s[k]$ is obtained by the summation over M statistically independent subcarriers, which results in a signal with non-constant envelope. The PAPR measure is an appropriate tool to quantify the impact of this phenomenon. In order to allow a fair comparison with OFDM, the PAPR is defined, as in [10], over only one symbol duration:

$$PAPR = \frac{\max_{k\in\{0,\ldots,N-1\}} |s[k]|^2}{\mathrm{E}\{|s[k]|^2\}}, \tag{4}$$

where E denotes the expectation. The PAPR is a random variable and a convenient technique to analyze its behavior is to compute its probability to exceed a given threshold, here denoted by γ. The CCDF gives this probability for every γ. So, our aim is now to find an approximation of the CCDF for the oversampled OFDM modulation. A similar computation has been already proposed for OFDM in [11]. Note also that in the rest of this paper, h is supposed to be a real-valued FIR filter with a unit $\mathcal{L}_2$ norm.

3.1. Analysis of a sample $s[k]$

The PAPR behavior is in direct relation with the statistical behavior of $s[k]$. Thus an important first point is to find the analytical expression for the variance of the real and imaginary parts of $s[k]$. Showing furthermore that they are equal.

Let $\sigma_R^2[k] = \mathrm{E}\{s_R[k]s_R[k]\}$ be the variance of the k-th sample of the real part signal s, denoted $s_R[k]$ with:

$$\begin{aligned} s_R[k] &= \sum_{m=0}^{M-1}\sum_{n\in\mathbf{Z}} c^R_{m,n} h[k-nN]\cos\left(\frac{2\pi}{M}m(k-\frac{D}{2})\right) \\ &\quad - c^I_{m,n} h[k-nN]\sin\left(\frac{2\pi}{M}m(k-\frac{D}{2})\right), \end{aligned} \tag{5}$$

where $c^R_{m,n}$ (resp. $c^I_{m,n}$) is the real (resp. imaginary) part of $c_{m,n}$. As $c_{m,n}$ is assumed to follow a zero-mean and a σ_c^2-variance process, $c^R_{m,n}$ and $c^I_{m,n}$ are zero-mean, with a $\frac{\sigma_c^2}{2}$ variance and are uncorrelated. Then by noting:

$$\begin{aligned} &A_{m,p,n,q,k} = \\ &\mathrm{E}\left\{c^R_{m,n}c^I_{p,q}\right\}\cos\left(\tfrac{2\pi}{M}m(k-\tfrac{D}{2})\right)\sin\left(\tfrac{2\pi}{M}p(k-\tfrac{D}{2})\right) \\ &+\mathrm{E}\left\{c^I_{m,n}c^R_{p,q}\right\}\cos\left(\tfrac{2\pi}{M}p(k-\tfrac{D}{2})\right)\sin\left(\tfrac{2\pi}{M}m(k-\tfrac{D}{2})\right), \end{aligned} \tag{6}$$

we finally obtain:

$$\begin{aligned} \sigma_R^2[k] &= \frac{\sigma_c^2}{2}M\sum_{n\in\mathbf{Z}} h[k-nN]^2 \\ &\quad -\sum_{m,p}\sum_{n,q} A_{m,p,n,q,k} h[k-nN]h[k-qN]. \end{aligned} \tag{7}$$

As the real and imaginary parts of $c_{m,n}$ are mutually uncorrelated, $A_{m,p,n,q,k} = 0$ and:

$$\sigma_R^2[k] = \frac{\sigma_c^2}{2}M\sum_{n\in\mathbf{Z}} h[k-nN]^2. \tag{8}$$

Similarly for the imaginary part of s, $s_I[k]$, we get:

$$\sigma_I^2[k] = \sigma_R^2[k] \triangleq \sigma_k^2. \tag{9}$$

In a second step, we prove that $s[k]$ follows a complex gaussian process. Indeed, at a given time index k, (1) shows that $s[k]$ is obtained by the summation over the M carriers of:

$$x_m = \sum_{n\in\mathbf{Z}} c_{m,n} h[k-nN] e^{j\frac{2\pi}{M}m(k-\frac{D}{2})}. \tag{10}$$

As $c_{m,n}$ are assumed to be uncorrelated, x_m are then uncorrelated too. In addition, it is easy to show that:

$$\mathrm{E}\{x_m\} = 0, \tag{11}$$

$$\sigma_x^2 = \mathrm{E}\{x_m x_m^*\} = \sigma_c^2\sum_{n\in\mathbf{Z}} h[k-nN]^2. \tag{12}$$

So, the mean and variance of x_m are both independent of m. Therefore, if we set $X_M = \sum_{m=0}^{M-1} x_m = s[k]$, it is clear that, for M large enough, based on the central limit theorem, X_M follows a complex gaussian process with a zero-mean and a variance given by $\sigma^2 = M\sigma_c^2\sum_{n\in\mathbf{Z}} h[k-nN]^2 = 2\sigma_k^2$. Thus s_R and s_I are jointly gaussian, as it can also be proved that they are uncorrelated, s_R and s_I are independent.

3.2. Approximation of the CCDF

Then, according to (9), we can also say that $|s[k]|$ follows a Rayleigh process and that $|s[k]|^2$ follows a χ^2 process with 2 degrees of freedom. Let $X = |s[k]|^2$, then the probability density function (pdf) of X is:

$$p_X(x) = \frac{1}{2\sigma_k^2}e^{-\frac{x}{2\sigma_k^2}}. \tag{13}$$

Let $Y = |s_0[k]|^2 = \frac{|s[k]|^2}{\mathrm{E}\{|s[k]|^2\}} = \frac{X}{\mathrm{E}\{|s[k]|^2\}}$. As h has a unit energy, it can be shown [5] that $\mathrm{E}\left\{|s[k]|^2\right\} = \frac{M}{N}\sigma_c^2$. By denoting α_k as follows:

$$\alpha_k = \frac{M}{2N}\frac{\sigma_c^2}{\sigma_k^2} = \frac{1}{N\sum_{n\in\mathbf{Z}} h[k-nN]^2}, \tag{14}$$

we can express the pdf of Y by:

$$p_Y(y) = \frac{M}{N}\sigma_c^2 p_X\left(\frac{M}{N}\sigma_c^2 y\right) = \alpha_k e^{-\alpha_k y}. \tag{15}$$

Then, for a given threshold γ, we have:

$$\Pr(|s_0[k]|^2 \le \gamma) = \int_{-\infty}^{\gamma} p_Y(y)dy = 1 - e^{-\alpha_k\gamma}. \tag{16}$$

Note that, as $\alpha_{k+pN} = \alpha_k$, we only need the N first terms. We also assume that the $|s_0[k]|^2$ samples are independant, so the CCDF is such that:

$$\Pr(PAPR \leq \gamma) = \Pr\left(\bigcap_{i=0}^{N-1} (|s_0[i]|^2 \leq \gamma)\right)$$
$$= \prod_{i=0}^{N-1} \Pr\left((|s_0[i]|^2 \leq \gamma)\right) = \prod_{i=0}^{N-1} \left(1 - e^{-\alpha_i \gamma}\right). \quad (17)$$

Finally, taking the complementary function, we find the following expression of the CCDF of the PAPR:

$$\Pr(PAPR \geq \gamma) \approx 1 - \prod_{i=0}^{N-1} \left(1 - e^{-\alpha_i \gamma}\right). \quad (18)$$

To illustrate the validity of our approximation, Fig. 1 shows the simulated CCDF and its approximation for $\eta = 3/2$ and different values of M. These results have been obtained with a prototype of length $6M$ obtained by the truncation and discretization of a Square Root of Raised Cosine (SRRC) function. The roll-off, r, of this SRRC prototype limits its frequency band in the range $|\nu| \leq (1+r)\frac{F_0}{2}$. As in [3], in order to fit with the time-frequency lattice of density $d = \eta^{-1}$, we set $r = \eta - 1$. It can be seen that, as expected from the central limit theorem, the quality of our approximation increases with M for a given η. That also a posteriori illustrates the usefulness of the independance assumption for the η values of practical interest ($\eta \leq 3/2$).

3.3. Influence of pulse shaping on the CCDF

In [11], it was shown that for OFDM the CCDF approximation only depends on the PAPR threshold γ. The result reported in (18) indicates that the situation is different for oversampled OFDM with a CCDF that also depends on a set of

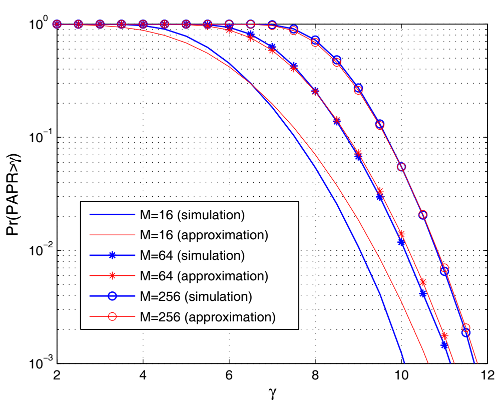

Fig. 1. Influence of the number of subcarriers M on the validity of the approximation ($\eta = 3/2$).

variables $\{\alpha_0, \ldots, \alpha_{N-1}\}$, where each α_k parameter is directly related to the prototype filter h (14) and also depends, for a given M, on the oversampling ratio. In this section, we assume that this ratio $\eta = N/M$ is fixed. The problem now is to find a prototype filter providing the minimum probability for the CCDF given in (18).

3.3.1. Some basic features

As the prototype filter h is of unit energy, it can be easily shown that:

$$\sum_{k=0}^{N-1} \sum_{n \in \mathbf{Z}} h[k - nN]^2 = 1. \quad (19)$$

Consequently, according to (14), we have:

$$\sum_{k=0}^{N-1} \frac{1}{\alpha_k} = N. \quad (20)$$

Let $u[k] = \sum_k h[k - nN]^2$ with $k = 0, \ldots, N-1$, it is obvious that we have: $u[k] \leq \sum_{k=0}^{N-1} u[k] = 1$. Then, our previous definition of α_k now writes as $\alpha_k = \frac{1}{Nu[k]}$. Consequently, the α_k's are lower bounded as follows:

$$\alpha_k \geq \frac{1}{N}. \quad (21)$$

3.3.2. Optimization of the CCDF

A prototype filter h that minimizes the CCDF given in (18) under constraint (20) has to be characterized by N parameters that solve the following minimization problem:

$$\min_{\alpha_i} \left(1 - \prod_{i=0}^{N-1} \left(1 - e^{-\alpha_i \gamma}\right)\right) \text{ st. } \sum_{i=0}^{N-1} \frac{1}{\alpha_i} = N. \quad (22)$$

This problem has already been tackled and solved in [9] for the OFDM/OQAM case. We have in this context the same optimization problem except that we have to determine N parameters, instead of M in [9]. We obtain that the necessary and sufficient optimality conditions for (22) are: $\forall i, \alpha_i = 1$, under the condition: $\gamma \leq u_0 N$[1]. Thus, the optimal CCDF for a given η is:

$$\Pr^{\text{opt}}(PAPR \geq \gamma) = 1 - \left(1 - e^{-\gamma}\right)^N. \quad (23)$$

We can then state the following theorem:

Theorem- Assuming that the $c_{m,n}$'s are i.i.d., for PAPR thresholds in the range $\gamma =]0, u_0 N]$, we can reach the optimal CCDF iff, for a given $\eta = N/M$, the prototype filter h satisfies:

$$\forall k \in \{0, \ldots, N-1\}, \sum_{n \in \mathbf{Z}} h[k - nN]^2 = \frac{1}{N}. \quad (24)$$

[1] $u_0 = 2 + LW\left(-2e^{-2}\right) \approx 1.59$, LW, the Lambert function, being an implicit function defined by: $LW(x)e^{LW(x)} = x$.

For a M carrier OFDM system the CCDF is approximated by $1-(1-e^{-\gamma})^M$ [11]. As for oversampled OFDM, we have $M < N$, the corresponding optimal CCDF is necessarily over the OFDM one for a given number of carriers.

4. A MEASURE OF CCDF QUALITY

Based on the optimality condition (24), we define a parameter ϵ_N such that:

$$\epsilon_N = \max_{k\in\{0,\ldots,N-1\}} \left| N \sum_{n\in\mathbf{Z}} h[k-nN]^2 - 1 \right|. \qquad (25)$$

For a given prototype h, ϵ_N corresponds to a measure of distance to CCDF optimality, i.e. if for a given $\eta = N/M$, $\epsilon_N = 0$ then h has an optimal CCDF. We have computed ϵ_N for different types of prototypes: the SRRC filters computed as indicated in section 3.2 and the perfectly orthogonal filters given in [6]. In this latter case, the prototype length is such that given two integer parameters N_0, M_0 satisfying $M_0 N = M N_0 = lcm(M,N)$ we have $L_h = m N_0 M$ with m an integer. The ϵ_N values, together with the CCDF curves derived from (18), are presented in Fig. 2 for $M = 1024$. We can note the following points concerning the CCDF of the oversampled OFDM:

- As already mentioned, in this case the optimal CCDF is necessarily over the one of OFDM, the difference decreases with η;
- For the SRRC prototype, the CCDF is practically independent of L_h. For a given η (here 3/2 or 5/4), this prototype provides the best CCDF but the resulting MCM system is not perfectly orthogonal ;
- Differently from what has been noted for OFDM/OQAM [9], orthogonality does not imply here CCDF optimality, a result which can be also deduced from an analysis of (24) and [6, (23)]. Furthermore the results also depend on the optimization criterion of the prototype. Being approximately at 1.5 dB of the optimal distribution for $\eta = 3/2$, the time-frequency localization (Loc.) criterion leads to the worst result. For the energy criterion, it can be noted that, as shown for $\eta = 3/2$, increasing L_h yields a better CCDF.

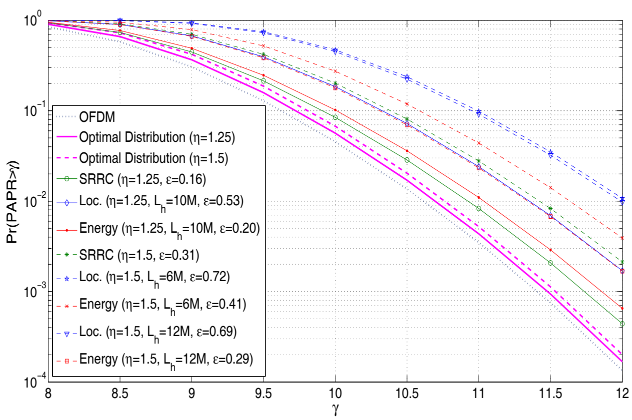

Fig. 2. Impact of different prototype filters and oversampling ratios on ϵ_N and on the CCDF curves.

5. CONCLUSION

Our analysis of the PAPR for oversampled OFDM has led to an approximate expression of its CCDF. We have also provided the necessary and sufficient conditions that prototype filters have to satisfy to provide an optimal CCDF for a given oversampling ratio. In the present case, differently from OFDM and OFDM/OQAM, orthogonality of the prototype does not imply optimality of the CCDF. Therefore an extension of the design method in [6] has to be derived to get orthogonal prototypes satisfying the CCDF constraints.

6. REFERENCES

[1] T. Cooklev, *Wireless communication standards*, IEEE standards wireless networks series. IEEE Press, New York, 2004.

[2] H. Boelcskei, "Efficient design of pulse shaping filters for OFDM systems," in *SPIE Conference*, July 1999.

[3] R. Vallet and K. Haj Taieb, "Fraction spaced multi-carrier modulation transmission," *Wireless Personal Communications*, vol. 2, pp. 97–103, 1995.

[4] R. Hleiss, P. Duhamel, and M. Charbit, "Oversampled OFDM Systems," in *DSP '97*, July 1997.

[5] C. Siclet, *Application de la théorie des bancs de filtres à l'analyse et à la conception de modulations multiporteuses orthogonales et biorthogonales*, Ph.D. thesis, 2002.

[6] C. Siclet, P. Siohan, and D. Pinchon, "Perfect Reconstruction Conditions and Design of Oversampled DFT Modulated Transmultiplexers," *to appear in JASP*.

[7] L. Hanzo, M. Münster, B. J. Choi, and T. Keller, *OFDM and MC-CDMA for broadband Multi-Users communications, WLANs and Broadcasting*, IEEE press. Wiley Series, 2003.

[8] S. Ben Slimane, "Peak to Average Power Ratio Reduction of OFDM Signals using Pulse Shape," in *Globecom '00*, November 2000, pp. 1412 – 1416.

[9] A. Skrzypczak, P. Siohan, and J. P. Javaudin, "Analysis of the Peak-to-Average Power Ratio for OFDM/OQAM," *submitted.*

[10] R. W. Bäuml, R. F. H. Fischer, and J. B. Huber, "Reducing the Peak-to-Average Power Ratio of Multicarrier Modulation by Selected Mapping," *Electronics Letters*, vol. 32, no. 22, pp. 2056–2057, October 1996.

[11] R. van Nee and A. de Wild, "Reducing the Peak-to-Average Power Ratio of OFDM," in *VTC*, May 1998, pp. 2072–2076.