UNIFORM THRESHOLD SCALAR QUANTIZER PERFORMANCE IN WYNER-ZIV CODING WITH MEMORYLESS, ADDITIVE LAPLACIAN CORRELATION CHANNEL

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ABSTRACT

The performance of a uniform-threshhold scalar quantizer in Wyner-Ziv coding is investigated in this paper. To derive analytical expressions we assume the abstract correlation channel from the side information to the source to be encoded is memoryless, additive Laplacian. Furthermore, in order to focus our attention on the performance of the quantizer, the Wyner-Ziv coding scheme is assumed to encode the quantizer output by using perfect Slepian-Wolf coding. Analytical expressions for the operational rate-distortion function are obtained for this case. By evaluating these analytical expressions, we show that scalar quantization with a mid-tread uniform threshhold quantizer, followed by perfect Slepian Wolf coding achieves performance which is close to the theoretical Wyner-Ziv rate-distortion bound at low rates.

1. INTRODUCTION

In this paper we describe a scalar quantizer Q defined on the real line by two sequences of real numbers: $\{x_k\}_{k=-\infty}^{\infty}$ and $\{y_k\}_{k=-\infty}^{\infty}$. To quantize a real number x, the quantizer Q simply finds an index $k^* = Q(x)$ such that $x_{k^*} < x \leq x_{k^*+1}$, and reconstructs x as y_{k^*}. For this reason, $\{x_k\}$ are called threshold levels, and $\{y_k\}$ are called reconstruction levels. In a special case when there exists a constant $\Delta > 0$ such that $x_{k+1} - x_k = \Delta$ for all k, the quantizer Q is called a uniform threshold quantizer. If in addition $x_k = k\Delta$, Q is called a uniform threshold symmetric quantizer.

If Q is applied to a random variable X, its performance can be analyzed by the rate

$$R_Q \stackrel{\Delta}{=} H(Q(X)), \tag{1}$$

where $H(Q(X))$ denotes the entropy of the random variable $Q(X)$, and the average distortion D_Q incurred in the quantization process. In most practical applications, the distortion of interest is the mean square error, i.e.,

$$D_Q \stackrel{\Delta}{=} \mathbf{E}(X - Q^{-1}(Q(X)))^2, \tag{2}$$

where the inverse mapping $Q^{-1}(k) \stackrel{\Delta}{=} y_k$ for any integer k, and $\mathbf{E}$ stands for standard expectation. Throughout this paper D_Q refers to mean square error unless specified otherwise.

The rate-distortion performance of uniform threshold quantizers for a single source has been studied extensively in the literature of traditional lossy source coding. In this paper we focus our discussion on memoryless Laplacian sources, the choice of which is justified by the fact that these sources are often used to model innovation processes in applications like audio, image, and video compression. For memoryless Laplacian sources, uniform threshold quantizers were shown in [3] to satisfy the necessary conditions for optimality. In the same paper, Berger also provided parametric equations to analytically calculate R_Q and D_Q of a uniform threshold *symmetric* quantizer Q for a zero-mean Laplacian source. Farvardin took a numerical approach [2] and showed that for a *zero-mean* Laplacian source, at rate 1 bit per sample the distortion yielded by an optimum uniform threshold quantizer is within 0.86dB of the theoretical limit given by the distortion-rate function of the same Laplacian source. Note here that the optimum uniform threshold quantizer considered in [2] is not symmetric. In fact, if we are constrained to use uniform threshold symmetric quantizers, the gap at rate 1 bit per sample becomes 3.61dB [1], and these quantizers cannot work under 1 bit per sample [2, 3]. In this paper, we derive parametric expressions for R_Q and D_Q when Q is a uniform threshold quantizer but not symmetric. These expressions are provided in Section 2. Using these expressions we can calculate D_Q for all rates R_Q and with better precision than the numerical approach [2].

Beyond traditional lossy source coding, more specifically, in Wyner-Ziv source coding [5] (also called lossy source coding with side information at the decoder), very little is known about the performance of uniform threshold quantizer. Contrary to this fact, in applications like distributed source coding or asymmetric video compression where Wyner-Ziv source coding is preferred over traditional lossy source coding because it promises a low-complexity encoder, uniform threshold quantizer is often a natural choice due to its simplicity and low computational complexity. In view of these, Section 3 of this paper is thus devoted to analyzing the performance of uniform threshold quantizers in Wyner-Ziv coding. In particular, we consider the following memoryless model. Let W, and N denote two independent Laplacian random variables, and X be another random variable such that

$$X = W + N. \tag{3}$$

Let $\{(X_i, W_i, N_i)\}_{i=1}^{\infty}$ be a sequence of independent copies of (X, W, N). Assume that $\{W_i\}_{i=1}^{\infty}$ is available only at the decoder as the side information. Wyner and Ziv showed that the rate-distortion function $R_{X|Y}^{WZ}(D)$ of the source $\{X_i\}_{i=1}^{\infty}$ given the side information $\{W_i\}_{i=1}^{\infty}$ is

$$R_{X|W}^{WZ}(D) \stackrel{\Delta}{=} \inf_U [I(X;U) - I(W;U)] \tag{4}$$

where the infimum is taken over the set of all random variables U taking values in an arbitrary finite set $\mathcal{U}$ such that $U \to X \to W$ is a Markov chain, and there exists a function $f : \mathcal{U} \times \mathcal{R} \to \hat{\mathcal{R}}$ such that

$$\mathbf{E}(X - f(U, W))^2 \leq D, \tag{5}$$

where $\mathcal{R}$ denotes the set of all real numbers and the $\hat{\mathcal{R}}$ denotes the reproduction alphabet. In order to focus our attention on the performance of the quantizer Q, we assume that the quantizer output $\{Q(X_i)\}_{i=1}^{\infty}$ will be encoded and decoded by using a perfect

Slepian-Wolf code [6]. Hence the rate R_Q is given by

$$R_Q \stackrel{\Delta}{=} H(Q(X)|W),$$

where $H(Q(X)|W)$ denotes the conditional entropy of $Q(X)$ given W. The distortion D_Q can be calculated by

$$D_Q = \mathbf{E}(X - Q^{-1}(Q(X)|W))^2,$$

where it follows from (5) that the inverse mapping Q^{-1} depends on W, i.e., for each realization w of W, the quantizer Q might have a different sequence of reconstruction levels $\{y_k\}$. In this paper for any realization w of W, we derive analytical expressions of $H(Q(X)|w)$ and $\mathbf{E}(X - Q^{-1}(Q(X)|w))^2$ by investigating the performance of uniform threshold quantizer for a Laplacian source with non-zero mean. We then evaluate these expressions to determine R_Q and D_Q for a memoryless Laplacian side information coupled with a memoryless, additive Laplacian correlation channel. Our results show that at rate 0.5 bits per sample, the distortion given by uniform quantization followed by perfect Slepian-Wolf coding is within 1dB of the theoretical limit from (4).

2. UNIFORM QUANTIZATION FOR LOSSY SOURCE CODING OF A LAPLACIAN SOURCE

Consider the one-dimensional quantization of a discrete-time memoryless zero-mean stationary process (source) $\{X_i\}_{i=1}^{\infty}$ with marginal pdf $p(x) = \frac{\lambda}{2}e^{-\lambda|x|}$ and variance $\sigma_X^2 = \mathbf{E}[X_1^2] = \frac{2}{\lambda^2}$. Consider a uniform threshold quantizer Q given by $\{x_k\}_k$ and $\{y_k\}_k$. Throughout this section we drop the subscript Q in R_Q and D_Q defined in (1) and (2), respectively. From (1) and (2) we see that an optimum quantizer Q minimizes

$$D = \sum_k \int_{x_k}^{x_{k+1}} p(x)(y_k - x)^2 dx. \qquad (6)$$

subject to the constraint $R = -\sum_k p_k \log_2 p_k$, where

$$p_k = \int_{x_k}^{x_{k+1}} \frac{\lambda}{2} e^{-\lambda|x|} dx. \qquad (7)$$

For a given set of threshhold levels, the reconstruction levels that minimize mean-squared error are given by:

$$y_k = \frac{1}{p_k} \int_{x_k}^{x_{k+1}} x p(x) dx. \qquad (8)$$

As shown in [3], necessary conditions for optimal scalar quantizers which minimize $J = R + \Lambda D$ are given by the following set of nonlinear equations:

$$p_{k+1} = p_k \exp\left[\Lambda(y_{k+1} - y_k)(y_{k+1} + y_k - 2x_k)\right], \qquad (9)$$

Further, [3] shows that for a Laplacian source, uniform threshhold quantizers ($x_k = k\Delta + \epsilon$) satisfy (9). For symmetric uniform threshhold quantizers ($\epsilon = 0$), the entropy rate and quantizer distortion are given by the following set of parametric equations:

$$\begin{aligned} R &= 1 + (1-\theta)^{-1}[-\theta \log_2 \theta - (1-\theta)\log_2(1-\theta)] \\ &= 1 + (1-\theta)^{-1} H(\theta), \qquad (10) \end{aligned}$$

$$D = \lambda^{-2}[1 - \theta \frac{\ln^2 \theta}{(1-\theta)^2}] \qquad (11)$$

where $H(\cdot)$ denotes the binary entropy function and $\theta = e^{-\lambda\Delta}$.

Noting that a symmetric uniform threshhold quantizer with an even number of levels can never yield an entropy rate below $R = 1$ bit per symbol, we derive the rate and distortion expressions for uniform threshhold quantizers with $x_k = \Delta k + \varepsilon$.

For $k \geq 0$ we have:

$$p_k = \int_{x_k}^{x_{k+1}} \frac{\lambda}{2} e^{-\lambda|x|} dx = e^{-\lambda\varepsilon} e^{-\lambda k \Delta}[1 - e^{-\lambda\Delta}]/2. \qquad (12)$$

$$\begin{aligned} y_k &= \frac{1}{p_k} \int_{x_k}^{x_{k+1}} x \frac{\lambda}{2} e^{-\lambda|x|} dx \\ &= \varepsilon + \frac{1}{\lambda} + k\Delta - \Delta \frac{e^{-\lambda\Delta}}{1 - e^{-\lambda\Delta}}. \qquad (13) \end{aligned}$$

It is clear from symmetry that $y_{-1} = 0$. For p_{-1} we have:

$$\begin{aligned} p_{-1} &= \frac{\lambda}{2}\left[\int_{-\Delta+\varepsilon}^{0} e^{\lambda x} dx + \int_0^{\varepsilon} e^{-\lambda x} dx\right] \\ &= \frac{2 - e^{\lambda(\varepsilon-\Delta)} - e^{-\lambda\varepsilon}}{2}. \qquad (14) \end{aligned}$$

Then quantizer distortion is given by:

$$D = \frac{2}{\lambda^2} - \sum_k {y_k}^2 p_k. \qquad (15)$$

For the case where $\varepsilon = 0.5\Delta$, it can be shown that (12–15) yield:

$$R = H(\sqrt{\theta}) + \sqrt{\theta}(1 - \log_2(1-\theta)) - \sqrt[3]{\theta} \frac{\log_2 \theta}{1-\theta}, \quad (16)$$

$$D = \lambda^{-2}[2 - \sqrt{\theta}[(1 - 0.5\ln\theta)^2 + \theta \frac{\ln^2\theta}{(1-\theta)^2}]]. \qquad (17)$$

Due to page limit, detailed derivations of (16–17) and the other results in the next section are omitted.

One can verify that there is a perfect correspondence between analytical quantizer performance (16, 17) and numerical ones. Further, the performance of this quantizer is very close to the rate-distortion performance bound for low rates as described in [2]—Table 1 compares SNR (signal-to-noise ratio) values (SNR = $10\log_{10}\frac{\sigma_x^2}{D}$) for uniform scalar quantization with the theoretical rate-distortion bound at various bit-rates. In the next section, we will derive rate-distortion performance expressions for the case of Wyner-Ziv coding with a Laplacian channel model, with the intent of characterizing the corresponding performance gap.

R (bps)	R(D)	Analytical Performance
1	6.62	5.767
2	12.66	11.326
3	18.68	17.208

Table 1. Comparison of the analytical SNR (in dB) of a uniform threshhold scalar quantizer versus theoretical source-coding bound for a Laplacian source.

3. UNIFORM QUANTIZATION IN WYNER-ZIV CODING

In this section we derive expressions for the entropy-rate and quantizer distortion of a scalar quantizer when used for Wyner-Ziv coding, where the correlation channel from the side information to the source is given by (3). Assume that the side-information W has distribution $p(w) = \frac{\alpha}{2}e^{-\alpha|w-w_0|}$, and the channel noise N is independent of W with distribution $p(n) = \frac{\lambda}{2}e^{-\lambda|n|}$. As discussed above in the introduction, we assume a Wyner-Ziv coding scheme that encodes the quantizer output by using perfect Slepian-Wolf coding. To this end we will first derive performance expressions for quantization of a non-zero mean Laplacian source with a symmetric uniform threshhold quantizer.

3.1. Performance of Uniform Threshold Quantizer for Laplacian Source with Non-Zero Mean

We investigate the performance of a symmetric uniform threshold quantizer with intervals $x_i = i\Delta$ for a Laplacian source given by $f(x) = \frac{\lambda}{2}e^{-\lambda|x-x_0|}$ with mean $x_0 = \varepsilon\Delta$, where $0 \le \varepsilon < 1$.

We have three cases here: $i \ge 1$, $i \le -1$ and $i = 0$. For the case of $i \ge 1$ we can write:

$$p_i = \int_{x_i}^{x_{i+1}} \frac{\lambda}{2}e^{-\lambda(x-x_0)}dx = \frac{e^{\lambda x_0}e^{-\lambda i\Delta}[1-e^{-\lambda\Delta}]}{2}, \quad (18)$$

$$y_i = \frac{1}{p_i}\int_{x_i}^{x_{i+1}} \frac{\lambda}{2}e^{-\lambda(x-x_0)}dx = \Delta i + \frac{1}{\lambda} - \Delta\frac{e^{-\lambda\Delta}}{1-e^{-\lambda\Delta}}. \quad (19)$$

For the case of $i \le -1$ we have:

$$\begin{aligned} p_i &= \frac{e^{-\lambda x_0}e^{\lambda i\Delta}[e^{\lambda\Delta}-1]}{2}, \text{ and} && (20)\\ y_i &= \Delta i - \frac{1}{\lambda} + \Delta\frac{e^{\lambda\Delta}}{e^{\lambda\Delta}-1}. && (21)\end{aligned}$$

For the case of $i = 0$ we find:

$$\begin{aligned} p_0 &= \int_0^{\varepsilon\Delta}\frac{\lambda}{2}e^{\lambda(x-\varepsilon\Delta)}dx + \int_{\varepsilon\Delta}^{\Delta}\frac{\lambda}{2}e^{-\lambda(x-\varepsilon\Delta)}dx\\ &= \frac{1}{2}[2 - e^{-\lambda\varepsilon\Delta} - e^{\lambda(\varepsilon-1)\Delta}], \text{ and} && (22)\\ y_0 &= \frac{e^{-\lambda\varepsilon\Delta} - (\lambda\Delta+1)e^{\lambda(\varepsilon-1)\Delta} + 2\lambda\varepsilon\Delta}{\lambda(2-e^{-\lambda\varepsilon\Delta}-e^{\lambda\Delta(\varepsilon-1)})}. && (23)\end{aligned}$$

The quantizer distortion is given by

$$\begin{aligned} D_\epsilon &= \sum_i \int_{x_i}^{x_{i+1}}(x-y_i)^2 f(x)dx\\ &= \frac{2}{\lambda^2} + \Delta^2\varepsilon^2 - \sum_i y_i^2 p_i. && (24)\end{aligned}$$

Then it can be shown that

$$\begin{aligned} A1 &\triangleq \sum_{i=1}^{\infty} y_i^2 p_i\\ &= \frac{\theta^{-\varepsilon}}{2\lambda^2}[1 + \theta\frac{\ln^2\theta}{(1-\theta)^2} - \frac{(1-\theta(1-\ln\theta))^2}{1-\theta}], && (25)\end{aligned}$$

and

$$A3 \triangleq \sum_{i=-\infty}^{-1} y_i^2 p_i = \frac{\theta^{\varepsilon}}{2\lambda^2}[1 + \theta\frac{\ln^2\theta}{(1-\theta)^2}]. \quad (26)$$

Taking into consideration (22) and (23), A2 is given by:

$$\begin{aligned} A2 &\triangleq y_0^2 p_0\\ &= \frac{[\theta^{\varepsilon} + (\ln\theta - 1)\theta^{1-\varepsilon} - 2\varepsilon\ln\theta]^2}{2\lambda^2[2-\theta^{\varepsilon}-\theta^{1-\varepsilon}]}. && (27)\end{aligned}$$

Thus the quantizer distortion D can be computed as follows:

$$D_\epsilon = \frac{2}{\lambda^2} + \Delta^2\varepsilon^2 - A1 - A2 - A3. \quad (28)$$

Let us compute rate $R_\epsilon = -\sum_{-\infty}^{\infty} p_i \log_2 p_i$. We start with $J3 \triangleq -\sum_{i=1}^{\infty} p_i \log_2 p_i$:

$$\begin{aligned} J3 &= -\sum_{i=1}^{\infty}\frac{e^{\lambda\Delta\varepsilon}e^{-\lambda i\Delta}[1-e^{-\lambda\Delta}]}{2}\Big[\lambda\Delta\varepsilon\log_2 e - 1\\ &\quad - \lambda i\Delta\log_2 e + \log_2[1-e^{-\lambda\Delta}]\Big]\\ &= \frac{\theta^{-\varepsilon}}{2}\Big[1 + \frac{H(\theta)}{(1-\theta)}\\ &\quad + (1-\theta)[-1+\log_2(1-\theta)] + \varepsilon\theta\log_2\theta\Big]. && (29)\end{aligned}$$

Given (22) and (23), $J2 \triangleq -p_0\log_2 p_0$ can be written as:

$$J2 = -\frac{1}{2}[2-\theta^{\varepsilon}-\theta^{1-\varepsilon}][\log_2(2-\theta^{\varepsilon}-\theta^{1-\varepsilon}) - 1]. \quad (30)$$

Let us find $J1 \triangleq -\sum_{i=-\infty}^{-1} p_i\log_2 p_i$:

$$\begin{aligned} J1 &= -\sum_{i=-\infty}^{-1} p_i\log_2 p_i && (31)\\ &= e^{-\lambda\Delta\varepsilon}\Big[-\sum_{i=-\infty}^{-1}\frac{e^{\lambda i\Delta}(e^{\lambda\Delta}-1)}{2}[-1+\lambda\Delta i\log_2 e && (32)\\ &\quad + \log_2(e^{\lambda\Delta}-1)] + \lambda\Delta\varepsilon\log_2 e\sum_{-\infty}^{-1}\frac{e^{\lambda i\Delta}(e^{\lambda\Delta}-1)}{2}\Big]\\ &= \frac{\theta^{\varepsilon}}{2}[1 + (1-\theta)^{-1}H(\theta) - \varepsilon\log_2\theta]. && (33)\end{aligned}$$

3.2. Performance of Uniform Threshhold Quantizer in Wyner-Ziv Coding

Based on the rate-distortion expressions derived above, we can now derive expressions for the performance of uniform-threshhold scalar quantizers used for Wyner-Ziv coding. We assume a Wyner-Ziv coding scheme consisting of uniform scalar quantization followed by perfect Slepian-Wolf coding. Consider the symmetric uniform threshhold quantizer with $x_i = i\Delta$, $i \in \mathbb{Z}$ considered above. As described in the introduction, we will consider a Laplacian distributed side information, and consider a Wyner-Ziv channel which adds independent, zero-mean Laplacian noise. In this setting the actual value of the side information, denoted w, will 'offset' the mean of the source distribution. The rate-distortion expressions for a fixed value of the side information are given by (29–33) and (24) with $\epsilon = \frac{w}{\Delta} - \lfloor\frac{w}{\Delta}\rfloor$.

Noting that $p(w) = \frac{\alpha}{2}e^{-\alpha|w-w_0|}$, we see that the distribution of ϵ is $p(\epsilon) = \sum_{i\in\mathbb{Z}}\frac{\alpha}{2}e^{-\alpha|i\Delta+\epsilon\Delta-w_0|}$. Thus, the average rate and average distortion for the Wyner-Ziv case are,

$$\begin{aligned} R_{WZQ} = \mathbf{E}_\epsilon[R_\epsilon] &= \sum\int_0^1 R_\epsilon\frac{\alpha}{2}e^{-\alpha|i\Delta+\epsilon\Delta-w_0|}d\epsilon && (34)\\ D_{WZQ} = \mathbf{E}_\epsilon[D_\epsilon] &= \sum\int_0^1 D_\epsilon\frac{\alpha}{2}e^{-\alpha|i\Delta+\epsilon\Delta-w_0|}d\epsilon && (35)\end{aligned}$$

The rate and distortion expressions for the Wyner-Ziv case can be computed by performing the integration in (34), (35) numerically.

3.3. Performance Evaluation

In this section, we present numerical results for the rate-distortion performance of uniform scalar quantizers used for Wyner-Ziv coding for the model described in the introduction. We consider a zero-mean, unit-variance Laplacian side-information for all simulations reported here, with the channel modeled as addition of zero-mean, independent, additive Laplacian noise with variance $\sigma_n^2 = 0.1$.

R (bps)	Simulated	Analytical Performance
1	5.09	5.09
2	11.12	11.12
3	17.16	17.15

Table 2. Comparison of the analytical and simulated SNR (in dB).

First we verify the correctness of the rate-distortion expressions given by (34) and (35). To this end, we simulate Wyner-Ziv coding using scalar quantization and perfect Slepian-Wolf coding, for blocks of 10^5 source vectors generated for each of a range of side-information values, governed by the considered model. Table 2 compares the rate-SNR performance (SNR $\triangleq 10\log_{10}\frac{\sigma_n^2}{D}$) obtained by simulation with the rate-distortion performance obtained by evaluating equations (34) and (35). As can be seen, the analytical results are in good agreement with the results obtained by simulation.

Next we evalute the performance of various uniform threshhold quantizers $\mathcal{Q}_l$ for Wyner-Ziv coding. The quantizers have reconstruction levels of the form $\mathcal{Q}_l : x_k = k\Delta + \gamma_l$, and are thus characterized by γ_l which is the offset of x_0 from the origin. Note that quantizing a zero-mean random variable using quantizer Q_l is equivalent to quantizing a random variable with mean γ_l using quantizer Q_0. Thus, (34) and (35) can be used to obtain the rate-distortion performance for the case in question. Figure 1 shows the rate-SNR curves for quantizers with $\gamma = 0.05\Delta, 0.25\Delta, 0.5\Delta$. As can be seen, the performance of the midtread quantizer (i.e. the quantizer with $\gamma = 0.5\Delta$) dominates the performance of the other quantizers. This result is satisysing, as it is analogous to the result obtained for traditional source coding.

Finally, we are now in a position to characterize the performance loss of uniform scalar quantization followed by perfect Slepian-Wolf coding. In order to do this, we compare the rate-distortion performance given by (34), (35) for a mid-tread uniform threshold quantizer with the Wyner-Ziv rate distortion bound [5]. The Wyner-Ziv rate-distortion bound is evaluated numerically for the considered Wyner-Ziv model, using an extended Blahut-Arimoto algorithm [7]. Figure 2 shows the result of the comparison—as can be seen the performance of the uniform scalar quantizer is less than 1 dB away from the theoretical Wyner-Ziv bound at rates below 0.5 bits per symbol. At high rates, the performance of the uniform scalar quantizer is, asymptotically, 1.5 dB away from the Wyner-Ziv bound as predicted by source coding theory.

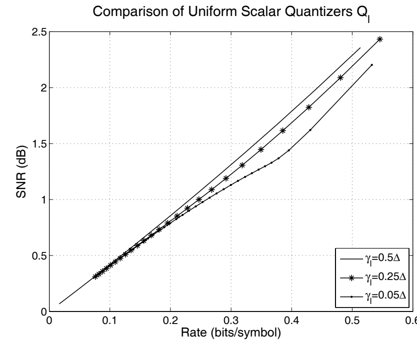

Fig. 1. Comparison of R-SNR curves for uniform threshhold quantizers with $\gamma_l = 0.05\Delta, 0.25\Delta, 0.5\Delta$.

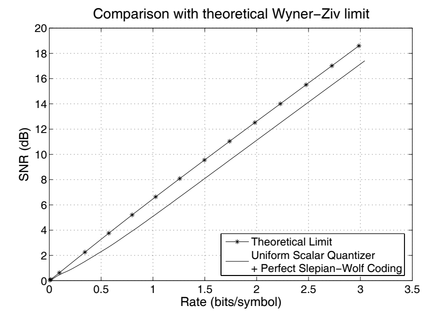

Fig. 2. Comparison of performance of mid-tread uniform scalar quantizer followed by perfect Slepian-Wolf coding, with theoretical Wyner-Ziv performance limit.

4. CONCLUSIONS

We have derived analytical expressions for rate-distortion performance of uniform threshold scalar quantizers for Wyner-Ziv coding where the correlation channel from the side information to the source is assumed to be memoryless, additive Laplacian. We have compared the performance of various uniform threshhold scalar quantizers for the problem at hand. By evaluating our analytical expressions, we have shown that at low bit-rates, mid-tread uniform threshold scalar quantizers followed by perfect Slepian-Wolf coding can yield performance which is close to the theoretical Wyner-Ziv bound.

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