OUTAGE PROBABILITY OF OSTBC: OPTIMAL TRANSMIT STRATEGY AND SUBOPTIMALITY OF ODD NUMBER OF TRANSMIT ANTENNAS

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ABSTRACT

Orthogonal space-time block codes (OSTBC) are an efficient mean in order to exploit the diversity offered by the wireless multiple-input multiple-output (MIMO) channel. It was shown that the Alamouti scheme, an OSTBC for $n_T = 2$ transmit antennas, achieves the capacity of such MIMO channels with $n_R = 1$ receive antenna. However, by increasing the number of transmit antennas, the transmission rate of the OSTBC is monotonically decreasing. Recently, this rate reduction was characterized completely.

Using a recent result on the properties of Gaussian quadratic forms, we show two key results: Assume the transmitter has not channel state information. We show that for given rate and SNR it is optimal to use a subset of all available antennas with equal power allocation. Further on, we show that an OSTBC for an odd number of transmit antennas is always outperformed by an OSTBC for either the next lower or the next higher even number of transmit antennas. Finally, the impact of spatial correlation with uninformed transmitter is characterized. We illustrate the theoretical results by numerical simulations. The theoretical and simulation results show the suboptimality of the OSTBC for an odd number of antennas with respect to the outage probability minimization.

1. INTRODUCTION

Recent information theoretic results have demonstrated that the ergodic capacity of a transmission system improves significantly with the use of multiple antennas (MIMO) [1, 2]. In contrast to the ergodic capacity which is a measure for the amount of average information that is error-free received, the outage probability is a more subtle measure for the probability of successful transmission while the channel is in a channel state. Since the instantaneous capacity depends on the channel state, it is itself a random variable. The first moment corresponds to the ergodic capacity. The cumulative distribution function (cdf) is the outage probability. The outage probability gives the probability that a given transmission rate cannot be achieved in one fading block.

Recently, the outage probability was studied for multiple antenna channel and space-time codes [3, 4]. The properties of the optimum transmission strategies change, if we replace the ergodic capacity as objective function with the outage probability. E.g. for no channel state information (CSI) at the transmitter, the optimum transmission strategy is to use only a fraction of the available number of transmit antennas. Telatar has already conjectured this in [1]. In [5], a part of this conjecture is verified. Furthermore, an algorithm which finds the optimum number of active antennas was proposed. In addition to this, in [5], a necessary and sufficient condition for the optimality of single-antenna processing was derived. In [6], the complete solution to the outage probability minimization for MISO systems was derived.

There has been a considerable amount of work on a variety of new codes and modulation signals, called space-time (ST) codes, in order to approach the huge capacity of such multiple antenna channels [1, 2]. One scheme of particular interest is the Alamouti scheme [7] for two transmit antennas. Later on, [8, 9] proposed more general schemes referred to as orthogonal space-time block codes (OSTBC) with the same properties as the Alamouti scheme like, e.g., a remarkably simple maximum-likelihood decoding algorithm. Interestingly, the combination of OSTBC with a MIMO antenna system can be represented equivalently as a single-inputsingle-output (SISO) system, where the channel gain is equal to the Frobenius norm of the actual MIMO channel. The performance of OSTBC with respect to mutual information was analyzed (among others) for the uncorrelated Rayleigh fading case in [10, 11] and for the more general case with different correlation scenarios and line of sight (LOS) components in [12].

Unfortunately, the Alamouti space-time code for two transmit and one receive antennas is the only OSTBC, which achieves the maximum possible mutual information of a MIMO system [1], since we cannot construct an OSTBC with a code rate equal one for more than two transmit antennas [8, 13]. Furthermore, by increasing the number of transmit antennas, the code rate of the OSTBC is monotonically decreasing. Recently, this rate reduction was characterized [13, 14]. In this work, we show the optimal transmit strategy that minimizes the outage probability for uninformed transmitter. Further on, based on the outage performance of OSTBC we show that an OSTBC for an odd number of transmit antennas, say k + 1 with even k, is always outperformed by an OSTBC for an even number of transmit antennas, either by k or k + 2, which has a rather interesting impact on the design and application of OSTBC.

2. SYSTEM MODEL

We consider a system with n_T transmit and n_R receive antennas. Our system model is defined by

$$Y = G_{n_T} P^{1/2} H + N , \qquad (1)$$

where G_{n_T} is the $(T \times n_T)$ transmit matrix, $Y = [y_1, \dots, y_{n_R}]$ is the $(T \times n_R)$ receive matrix, $H = [h_1, \dots, h_{n_R}]$ is the $(n_T \times$ n_R) channel matrix, P = diag(p) is the power allocation matrix and $N = [n_1, \ldots, n_{n_R}]$ is the complex $(T \times n_R)$ white Gaussian noise (AWGN) matrix, where an entry $\{n_{ti}\}$ of N $(1 \le i \le n_R)$ denotes the complex noise at the *i*th receiver for a given time t $(1 \le t \le T)$. The real and imaginary parts of n_{ti} are independent and $\mathcal{N}(0, n_T/(2\text{SNR}))$ distributed.

The channel matrix H for the case in which we have correlated transmit and correlated receive antennas is modeled as

$$\boldsymbol{H} = \boldsymbol{R}_{R}^{\frac{1}{2}} \cdot \boldsymbol{W} \cdot \boldsymbol{R}_{T}^{\frac{1}{2}}$$
(2)

with transmit correlation matrix $\mathbf{R}_T = \mathbf{U}_T \mathbf{\Lambda}_T \mathbf{U}_T^H$ and receive correlation matrix $\mathbf{R}_R = \mathbf{U}_R \mathbf{\Lambda}_R \mathbf{U}_R^H$ [15]. \mathbf{U}_T and \mathbf{U}_R are the matrices with the eigenvectors of \mathbf{R}_T and \mathbf{R}_R respectively, and $\mathbf{\Lambda}_T$, $\mathbf{\Lambda}_R$ are diagonal matrices with the eigenvalues of the matrix \mathbf{R}_T and \mathbf{R}_R , respectively. The random matrix \mathbf{W} has zero-mean iid complex Gaussian entries, i.e. $\mathbf{W} \sim \mathcal{CN}(0, \mathbf{I})$. We order the eigenvalues of the transmit covariance matrix \mathbf{P} in descending order, i.e. $p_1 \geq p_2 \geq ... \geq p_{n_T}$, as well as the eigenvalues of the channel correlation matrices \mathbf{R}_T and \mathbf{R}_R , i.e. $\lambda_1^T \geq \lambda_2^T \geq ... \geq \lambda_{n_T}^T$ and $\lambda_1^R \geq \lambda_2^R \geq ... \geq \lambda_{n_R}^R$.

2.1. Code construction and rate reduction

A space time block code is defined by its transmit matrix G_{n_T} given as

$$G_{n_T} = \sum_{j=1}^{q} \boldsymbol{A}_j \operatorname{Re}(s_j) + i \boldsymbol{B}_j \operatorname{Im}(s_j).$$
(3)

The symbols $\{s_j\}_{j=1}^q$ are elements of the vector $\boldsymbol{s} = [s_1, \ldots, s_q]^T$ with $s_1, \ldots, s_q \in \mathcal{C}$, where $\mathcal{C} \subseteq \mathbb{C}$ denotes a complex modulation signal set with unit average power, e.g. *M*-PSK. The $\boldsymbol{A}_j, \boldsymbol{B}_j, 1 \leq j \leq q$ are $T \times n_T$ matrices that satisfy the following properties:

$$\boldsymbol{C}_{j}\boldsymbol{C}_{j}^{H} = \boldsymbol{I}, \boldsymbol{C}_{j}\boldsymbol{C}_{k}^{H} = -\boldsymbol{C}_{k}\boldsymbol{C}_{j}^{H}.$$
(4)

In addition they satisfy the following property $A_j B_k^H = B_k A_j^H$

The code rate r_c of a space-time code is defined as $r_c = q/T$. For an OSTBC with n_T transmit antennas it was shown recently [14] that the maximum achievable rate is given by

$$r_c(n_T) = \frac{\lfloor \frac{n_T+1}{2} \rfloor + 1}{2\lfloor \frac{n_T+1}{2} \rfloor}$$
(5)

It is important to note, that $r_c(k+1) = r_c(k+2)$ with k even. Furthermore, it holds that $\lim_{n_T \to \infty} r_c(n_T) = 1/2$.

After matched filtering with $A_k^H H^H$ and $iB_k^H H^H$, respectively, the effective signal model induced by OSTBC is

$$\tilde{\boldsymbol{y}} = ||\boldsymbol{P}^{1/2}\boldsymbol{H}||_F \boldsymbol{s} + \tilde{\boldsymbol{n}}$$
(6)

with additive Gaussian noise \tilde{n} with i.i.d. entries.

2.2. Properties of Gaussian quadratic forms

We present the theorems from [6] for completeness. Consider the quadratic form $Q_n = \sum_{i=1}^n \lambda_i |z_i|^2$ where the real and imaginary part of z_i are i.i.d. normal distributed variables and λ_i are eigenvalues of some covariance matrix \boldsymbol{R} . \boldsymbol{R} is positive semidefinite, implying $\lambda_i \ge 0, 1 \le i \le n$. Note that $\mathbb{E}(Q_n) = \operatorname{tr}(\boldsymbol{R}) = \sum_{k=1}^n \lambda_k = 1$. Furthermore, let $\chi_d 2$ a random variable that is χ_2 distributed with d degrees of freedom. The main theorem proved in [6] is

Theorem 2.1

$$\inf_{\{Q_n \ge 0 | \mathbb{E}(Q_n) = 1\}} P(Q_n \le x) = \\ \begin{cases} P\{\frac{1}{d}\chi_d 2 \le x\}, & \forall x \in [x(d), x(d-1)], d = 1, 2, ..., n-1 \\ P\{\frac{1}{n}\chi_n 2 \le x\}, & \forall x \in [0, x(n-1)], \end{cases}$$

where $x(0) = \infty$ by definition.

In addition to this, the following corollaries are interesting

Corollary 2.1 If $x \ge 2$, and $\lambda^{(1)}$ majorizes $\lambda^{(2)}$, then

$$P\left\{\sum_{i=1}^{n} \lambda_i^{(1)} |z_i|^2 \le x\right\} \le P\left\{\sum_{i=1}^{n} \lambda_i^{(2)} |z_i|^2 \le x\right\},\$$

and conversely, if this inequality holds for all positive integers n and for any $\lambda^{(1)}$ that majorizes $\lambda^{(2)}$, then $x \geq 2$.

Corollary 2.2 If $x \leq 1$, and $\lambda^{(1)}$ majorizes $\lambda^{(2)}$, then

$$P\left\{\sum_{i=1}^{n} \lambda_i^{(1)} |z_i|^2 \le x\right\} \ge P\left\{\sum_{i=1}^{n} \lambda_i^{(2)} |z_i|^2 \le x\right\}.$$

3. OUTAGE PERFORMANCE

OSTBC have been analyzed from an average and outage performance point of view. Scenarios with delay constraints or where a certain level of reliability has to be guaranteed are better captured through an outage analysis. For coded systems, the outage probability itself serves as a lower bound on the frame error performance [16]. The outage probability P_{out} achievable with OSTBC is defined as the probability that the mutual information is smaller than a certain transmission rate R, i.e. with

$$P_{out}(R, n_T, \rho, r_c) = Pr(I \le R) , \qquad (7)$$

where I is the mutual information corresponding to the effective channel induced by OSTBC, given as

$$I = r_c \log \left(1 + \frac{1}{N_0} || \mathbf{P}^{1/2} \mathbf{H} ||_F 2 \right)$$

= $r_c \log \left(1 + \frac{1}{N_0} \sum_{j=1}^{n_T} p_j \sum_{i=1}^{n_R} |h_{j,i}|^2 \right)$ (8)

Note that the transmit and receive correlation is included in the statistics of the channel matrix H and its squared elements $|h_{j,i}|^2$ in (8). Using (8) in (7) results in

$$P_{out}(\rho, \tilde{R}_{n_T}, \boldsymbol{\lambda}^T, \boldsymbol{\lambda}^R, \boldsymbol{p}) = Pr\left(||\boldsymbol{P}^{1/2}\boldsymbol{H}||_F 2 \le N_0 \left(2^{\frac{R}{r_c}} - 1 \right) \right)$$
(9)

with the SNR $\rho = P/N_0$, the power allocation vector \boldsymbol{p} , and the effective rate $\tilde{R}_{n_T} = R/r_c(n_T)$.

3.1. Optimal power allocation

First, consider the following problem statement: The transmitter has no CSI and the channel is spatially uncorrelated. What is the optimal power allocation under sum transmit power constraint tr $P \leq P$?

Theorem 3.1 Let the transmitter have n_T transmit antennas and the effective transmission rate is \tilde{R}_{n_T} . Then it holds:

- The outage probability is minimized by equal power allocation over a subset of l antennas out of n_T .
- The SNR range in which only one antenna is active is given by

$$\underline{\rho} = \frac{P}{N_0} = \frac{2(2^{\bar{R}} - 1)}{-2\mathcal{L}_w(-1, -1/2\exp(-1/2)) - 1}.$$
 (10)

 \mathcal{L}_w is the Lambert W function. Its value for the parameters -1 and $-1/2 \exp(-1/2)$ approximately is -1.756. As a result, the single-antenna region can be written as $\underline{\rho} = \frac{2^{\tilde{R}}-1}{1.258}$.

In Theorem 2.1, it is shown that the minimum of the quadratic form is achieved for all regions [x(d), x(d-1)] by a $\chi 2$ distribution with d degrees of freedom. In our case, the range x(d) is parameterized by the SNR ρ . The optimality of the equal power allocation for high SNR values is proved in Corollary 2.2.

It follows that the outage probability in (9) as a function of l is given as¹

$$P_{out}(R, l, \rho) = 1 - \frac{\Gamma(ln_R, (2^{R/r_c(l)} - 1)\frac{l}{\rho})}{\Gamma(ln_R)}, \ 1 \le l \le n_T$$
(11)

3.2. Suboptimality of odd number of transmit antennas

The outage probability as a function of l in (11) is to be minimized with respect to l. The next lemma shows an important property of the transmit strategy that minimizes the outage probability.

Lemma 3.1 Consider even l and the three outage probability curves $P_{out}(R, l, \rho)$, $P_{out}(R, l + 1, \rho)$, and $P_{out}(R, l + 2, \rho)$. Denote the intersection points between the curves as $\rho_{l,l+1}$, $\rho_{l,l+2}$, and $\rho_{l+1,l+2}$. As the rate R scales from R to \tilde{R} , the SNR point ρ scales by a factor of

$$\tilde{\rho} = \rho \cdot \frac{2^{\tilde{R}} - 1}{2^{R} - 1}.$$
(12)

As a result, the order of the intersection points does not depend on R.

The following theorem shows that the outage probability for an odd number l + 1 of transmit antennas is always greater than the minimum of the outage probabilities for l or l + 2 transmit antennas. Assume $n_R = 1$ for simplicity. The result holds also for arbitrary n_R .

Theorem 3.2 Fix R and ρ . The minimum of the outage probability of the OSTBC

$$\min_{1 \le l \le n_T} P_{out}(R, l, \rho) = \min_{1 \le l \le n_T} 1 - \frac{\Gamma(l, (2^{R/r_c(l)} - 1)\frac{l}{\rho})}{\Gamma(l)} \quad (13)$$

is attained for even l or n_T .

Proof: Due to space limitations, we give here only the sketch of the proof. The complete proof can be found in [17].

Consider l even. Then the coding rates for l, l + 1, l + 2 are given by

$$r_c(l) = \frac{1}{2} + \frac{1}{l}$$
 and $r_c(l+1) = r_c(l+2) = \frac{1}{2} + \frac{1}{l+2}$.

By Theorem 3.1 and Corollary 2.2, we know that the SNR interval in which the intersection points of all outage probabilities with code rate one are, is given by

$$\frac{2^R - 1}{1.25} \le \rho \le 2^R - 1.$$

Denote the intersection points between the outage probability curves for l and l+1 as $\rho_{l,l+1}$, for l and l+2 as $\rho_{l,l+2}$, and for l+1 and l+2 as $\rho_{l+1,l+2}$. The intersection point $\rho_{l+1,l+2}$ lies in the interval that

is shifted as in (12) by the corresponding rate factor $2\frac{\frac{R}{2}+\frac{1}{l+2}-1}{2^{R}-1}$. Choose R = 1/2, then for all $l \ge 2$ we have $\rho_{l+1,l+2} \le 1$. Finally, it holds that for R = 1/2 and $\rho = 1$ the difference between the outage probability for l and l + 1 is negative, i.e. the intersection point $\rho_{l,l+1}$ is at a higher SNR than $\rho_{l+1,l+2}$. Then by Lemma 3.1 this holds for all rates and shows the suboptimality of l + 1 antennas.

The theorem says that an odd number of transmit antennas is always outperformed by either the lower or higher even number of transmit antennas. From an outage probability point of view, OS-TBC for an odd number of transmit antennas is strict suboptimal.

The suboptimality of the odd number of transmit antennas is explained in the following plastic way: The result in [6, Theorem 1] shows for MISO systems that the higher the SNR the more antennas are active in order to minimize outage probability. The impact of the code rate r_c as a function of the number of transmit antennas is as follows. The outage probability curve is shifted to the right the lower the code rate is. For even l, the code rate of l + 1 and l + 2 is equal. Therefore, the shift due to the code rate loss for l + 1 and l + 2 to the right is so large, that the diversity offered by l + 1 antennas does not yet come into effect.

3.3. Impact of spatial correlation for equal power allocation

Let us assume, that the transmitter is uninformed and the channel is spatially correlated. The transmitter does not know the channel correlation and cannot adapt to it. Regarding robustness against the worst case spatial correlation, it can be shown that equal power allocation is optimal. The following result characterizes the impact of spatial correlation on the outage probability of an OSTBC system. It is a corollary from Theorem 2 in [6].

Corollary 3.1 For fixed transmission rate R, the outage probability as a function of the correlation properties of the transmit antennas is characterized by the following statements: For SNR $\rho < \rho = \frac{2^R - 1}{2}$, the outage probability is a Schur-concave function of the channel covariance matrix eigenvalues λ , i.e. correlation decreases the outage probability

$$\boldsymbol{\lambda}^{1} \succ \boldsymbol{\lambda}^{2} \Longrightarrow P_{out}(\rho < \underline{\rho}, R, \boldsymbol{\lambda}^{1}, \mathbf{1}) \leq P_{out}(\rho < \underline{\rho}, R, \boldsymbol{\lambda}^{2}, \mathbf{1}),$$

For SNR $\rho > \overline{\rho} = 2^{2R} - 1$, the outage probability is a Schurconvex function of the channel covariance matrix eigenvalues λ , i.e. correlation increases the outage probability

$$\lambda^{1} \succ \lambda^{2} \Longrightarrow P_{out}(\rho < \overline{\rho}, R, \lambda^{1}, \mathbf{1}) \ge P_{out}(\rho < \overline{\rho}, R, \lambda^{2}, \mathbf{1}).$$

The corollary follows if one shifts the lower SNR by the higher coding rate max $r_c = 1$ to the left and if one shifts the higher SNR by the lowest coding rate min $r_c = \frac{1}{2}$ to the right, i.e. $\overline{\rho} = 2^{2R} - 1$.

¹We omitted the transmit and receive correlation argument for convenience.

4. NUMERICAL SIMULATIONS

4.1. Suboptimality of odd number of transmit antennas

In figure (1), the outage probability as a function of the SNR is shown for l = 2, 3, 4, 5, 6 active antennas and Rate R = 1. In figure



Fig. 1. Outage probability of OSTBC with $n_R = 1$ receive and $n_T = 2, 3, 4, 5, 6, R = 1$.

(1), the switching SNR points from two to three ρ_{23} , from two to four ρ_{24} , from four to five ρ_{45} , and from four to six ρ_{46} are shown. In order to minimize the outage probability we always choose the lowest of the curves. That means that the higher the SNR the more antennas are used. Up to 4.8 dB an orthogonal space-time coded system with two antennas is optimal. Then a system with four antennas is optimal in the range from 4.8 dB up to 5.4 dB. And from 5.6 dB a system with 6 antennas has minimum outage probability. Observe in figure (1), that as indicated in the proof of Theorem 3.2, the switching points from an even to an odd number of antennas ρ_{23} and ρ_{45} are at a higher SNR as the switching points from even to the next even number of antennas ρ_{24} and ρ_{46} , respectively.

4.2. Impact of spatial correlation

In figure (2), we compare an OSTBC for 6 and 10 transmit antennas in a correlated and uncorrelated scenario for rate R = 1. As shown



Fig. 2. Outage probability of OSTBC with $n_R = 1$ receive and $n_T = 6, 10, R = 1$ for different spatial correlations.

in Corollary 3.1, for small SNR values $\rho \leq \underline{\rho} \approx -3dB$ for R = 1, spatial correlation decreases the outage probability. For high SNR values $\rho \geq \overline{\rho} \approx 4.77dB$ for R = 1, spatial correlation increases the outage probability.

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