OPTIMUM ISI-FREE DMT SYSTEMS WITH INTEGER BITLOADING AND ARBITRARY DATA RATES: WHEN DOES ORTHONORMALITY SUFFICE?

Xuejie Song and Soura Dasgupta

Department of Electrical and Computer Engineering, The University of Iowa, Iowa City, USA.
Email: xsong,dasgupta@engineering.uiowa.edu

ABSTRACT

We consider the design of optimum ISI-free, discrete multitone transmission (DMT) systems designed to achieve a quality of service (QoS) requirement quantified by bit rate and symbol error rate specifications. The optimality is in the sense of minimizing the transmitted power given the QoS specifications subject to the knowledge of the channel and colored interference at the receiver input of the DMT system. Earlier papers, obtained their optimum DMT design based on approximations that included high bit rates and/or real (as opposed to integer) bit loading schemes, and showed that orthonormal transforms suffice for optimality. In this paper we relax both the high bit rate and the integer bitloading approximation, and provide sufficient conditions on the underlying modulation schemes for which orthonormal DMT remains optimum.

1. INTRODUCTION

Discrete multi-tone (DMT) modulation has proved to be an effective solution to the problem of reliable and efficient data transmission over frequency selective communication channels. It is a curent standard in various wireline applications like ADSL, VDSL, [1], and in the form of Orthogonal frequency division multiplexing (OFDM) has been proposed for fixed wireless standards like IEEE 802.11a. There has been recent interest in the optimal design of general DMT systems, [4]-[7]. In particular, in such wireline applications as ADSL and VDSL, where channel conditions do not undergo substantial changes after the initial setup, these channel conditions are fed back to the transmitter, and used to achieve optimum bit loading in a manner to be specified in the sequel. Recently several authors such as [4]-[8] have sought to further improve performance by using channel conditions to optimize the transmitter and receiver transforms. In doing so, unlike OFDM, they have used zero-padding redundancies.

Even though, OFDM it self uses orthonormal block transforms, [4], [6] and [8], in particular have studied whether any performance improvements occur if the orthonormality requirement is relaxed, by considering more general ISI-free systems, referred in the sequel as biorthogonal as opposed to orthonormal DMT. The conclusions of all three papers has been that relaxing orthonormality brings no performance improvements. The analysis in [4] and [6] makes two simplifying approximations. First they assume that the bit rate in each subchannel is high enough so that an asymptotic relation between SER and SNR may be utilized. When bit loading is used, certain subchannels may utilize very low bit rates, rendering this approximation unrealistic. Second is the equally unrealistic assmption that the bits assigned to each subchannel may be nointegers. While, [8] relaxes the high bit rate assumption it continues to assume potentially noninteger bitloading.

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In this paper, we remove both these assumptions and derive conditions on the SER-SNR relationship that ensures that orthonormality suffices for optimality. We also demonstrate by way of an example that PAM satisfies the sufficient conditions derived here. Our work contrasts with the work of [10], [11], where general, and indeed elegant characterizations, of joint transceiver optimization are given. The general framework in these papers does not answer the specific question of orthonormaility studied in this paper. As in [4], [6] and [8], we show that the sufficient condition we provide here also guarantees a separation principle described below.

Figure.1 depicts a baseband model of DMT communication system. The incoming data stream is converted into M-parallel data streams each operating at a rate that is M-times smaller than the original symbol rate. An M-point block orthogonal transform ($M \times M$ matrix G_0) is followed by a parallel to serial conversion, prior to transmission through the communication channel. The equalizer at the channel output is to keep the effective channel length, κ, small. To infuse resistance to channel induced intersymbol interference (ISI), extra redundancy of length κ is added at the channel input. This leads to a data rate reduction by a factor of $N = M + \kappa$ in each subchannel. After the equalizer one performs in succession the operations of redundancy removal ($M \times N$ matrix S_1), serial to parallel conversion, and an M-point inverse block orthogonal transform ($M \times M$ matrix S_0). In traditional OFDM, the two orthogonal transforms G_0, S_0 are Inverse discrete Fourier Transform (IDFT) and DFT operations respectively, and the redundancy takes the form of cyclic prefix. Recently more general block transforms, and the injection of zero padding redundancy , leading to *generalized DMT systems*, have been proposed [4]. It is such a zero-padding generalized DMT system that is the subject of this paper.

A DMT system is called *orthonormal* if ISI free transmission prevails, together with the requirement that the transform matrices G_0 and S_0 be unitary, i.e.

$$G_0^H G_0 = S_0^H S_0 = I.$$

By contrast this system is called *biorthogonal* is neither G_0 nor S_0 are restricted to be orthonormal, but are arbitrary nonsingular matrices, though the zero-ISI condition must still hold.

We consider here such a general biorthogonal DMT system supporting a service whose QoS requirement quantified in this paper by its bit rate and symbol error rate (SER) has been specified. Our goal is to select the input and output block transforms G_0 and S_0, the linear redundancy removal matrix S_1, and the number of bits/symbol assigned to each subchannel, in order to achieve the QoS specifications under a zero ISI condition with *minimum transmitted power*. We assume knowledge at the transmitter of the equalized channel characteristics and the second-order statistics of the noise at the equalizer output. *Such knowledge is implicit in bitloading algorithms currently employed over wireline applications such as VDSL.*

The sufficient condition provided here, also ensures the following separation principle: The optimum transceiver depends only on the knowledge of the channel and equalizer, and does not depend on the QoS requirements. The latter only affect bit loading. This *separation principle* has practical importance to such wireline applications as ADSL and VDSL, where once the the initial connection has been established, the channnel conditions do not change substantially, and at most suffer very slow drift. Thus once the estimates of the channel conditions have been fed back to the transmitter after the initial setup, for all practical purposes the transceiver does not have to be changed during the life of a given connection.

2. PROBLEM FORMULATION

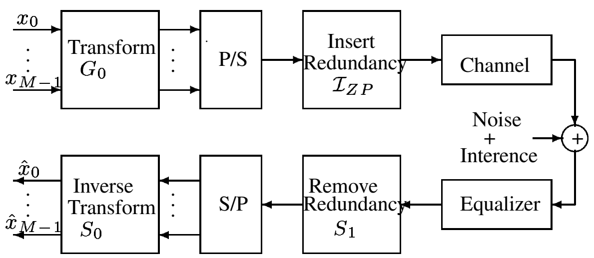

Fig. 1. DMT communication system.

Assume there are M data streams $x_0(n), x_1(n), \ldots, x_{M-1}(n)$ being fed into the DMT system. Assume that b_j is the number of bits/symbol assigned to the j-th subchannel, and the bit rate constraint imposed by the QoS requirement is that for some B,

$$\sum_{j=0}^{M-1} b_j = B \tag{2.2}$$

The QoS specifications stipulate that the SER in each subchannel shall not exceed η. Suppose the output signal to noise ratio (SNR) to achieve a SER of η, in a b-bits/symbol allocation, is

$$f(b) = \sigma_{\hat{x}}^2/\sigma_e^2 \tag{2.3}$$

where $\sigma_{\hat{x}}^2$ and σ_e^2 are the signal and noise power respectively. The function $f(\cdot)$ depends on the modulation scheme used. *The sufficient condition we seek is on* $f(.)$.

The equalized FIR channel is assumed known for transceiver design purpose, with length κ, it can be specified as

$$C(z) = c_0 + c_1 z^{-1} + \ldots + c_\kappa z^{-\kappa} \tag{2.4}$$

Define $N = M + \kappa$, call the $N \times N$ blocked version of $C(z)$, $\mathcal{C}(z)$,

$$\mathcal{C}(z) = \begin{bmatrix} c_0 & z^{-1}c_{N-1} & z^{-1}c_{N-2} & \ldots & z^{-1}c_1 \\ c_1 & c_0 & z^{-1}c_{N-1} & \ldots & z^{-1}c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{N-1} & c_{N-2} & \ldots & c_1 & c_0 \end{bmatrix} \tag{2.5}$$

with $c_i = 0$, for $i \geq \kappa$. With zero-padding redundancy insertion matrix expressed by

$$\mathcal{I}_{\mathcal{ZP}} = \begin{bmatrix} I_M \\ 0_{\kappa \times M} \end{bmatrix} \tag{2.6}$$

it can be verified that for some constant C_L,

$$C(z)\mathcal{I}_{\mathcal{ZP}} = C_L \tag{2.7}$$

Consider the singular value decomposition of C_L

$$C_L = U_c \begin{bmatrix} \Lambda_c \\ 0 \end{bmatrix} V_c^H = U_0 \Lambda_c V_c^H \tag{2.8}$$

where U_c and V_c are respectively $N \times N$ and $M \times M$ unitary matrices, Λ_c is a $M \times M$ real, positive definite diagonal matrix, and U_c is partitioned as $U_c = \begin{bmatrix} U_0 \\ U_1 \end{bmatrix}$, where U_0 is $N \times M$ and U_1 is $N \times \kappa$.

We assume ISI free transmission or the perfect reconstruction (PR) condition: $\hat{x}(n) = x(n)$ for all n, or

$$S_0 S_1 C_L G_0 = I \tag{2.9}$$

Then, given G_0, the class of all S_0, S_1 enforcing PR is completely characterized by

$$S_0 = G_0^{-1} \tag{2.10}$$

and

$$S_1 = V_c \Lambda_c^{-1} \begin{bmatrix} I_M & A \end{bmatrix} U_c^H \tag{2.11}$$

where A is an arbitrary $M \times \kappa$ matrix.

Under (2.9) $\sigma_{x_j}^2 = \sigma_{\hat{x}_j}^2$. Thus each subchannel is guaranteed to meet the SER of η if

$$\sigma_{x_j}^2 = f(b_j)\sigma_{e_j}^2 \tag{2.12}$$

Here $\sigma_{e_j}^2$ is the noise power at the output of the j-th subchannel. Thus total transmitted power is given by

$$P_B = \sum_{j=0}^{M-1} f(b_j)\sigma_{e_j}^2 [G_0^H G_0]_{jj} \tag{2.13}$$

Denote R_u to be the known autocorrelation matrix of the $N \times 1$ blocked vector $\mathbf{u}(n)$, of the noise at the equalizer output. Similarly R_w and R_v are the autocorrelation matrices of $\mathbf{w}(n)$, the $M \times 1$ blocked noise vector at the output of S_1, and $\mathbf{v}(n)$, $M \times 1$ blocked noise vector at the output of S_0, respectively. We have the relations

$$R_v = S_0 R_w S_0^H, \qquad R_w = S_1 R_u S_1^H \tag{2.14}$$

Then to meet the SER requirement the total transmitted power is given by

$$P_B = \sum_{j=0}^{M-1} f(b_j)[G_0^H G_0]_{jj}[S_0 R_w S_0^H]_{jj} \tag{2.15}$$

Thus, the optimization problem becomes: Given positive η, B, positive definite Hermitian R_u, and C_L, minimize (2.15) subject to (2.2) and (2.9), by selecting *nonnegative integer* b_j (bit loading), A (for selecting S_1) and the nonsingular matrix S_0.

3. INTERMEDIATE RESULTS

Because of (2.10), objective function P_B can be rewritten as

$$J(S_0) = \sum_{i=0}^{M-1} f(b_i)(e_i^T S_0 R_w S_0^H e_i)(e_i^T S_0^{-H} S_0^{-1} e_i) \tag{3.16}$$

with $f(b_i)$ being signal to noise power ratio and hence positive. Then the following theorem, see [6], characterizes a minimizing S_0.

Theorem 3.1 *Let the SVD of $M \times M$, positive definite Hermitian R_w be*

$$R_w = U\Lambda^2 U^H \tag{3.17}$$

with Λ real, diagonal and U unitary. Then for arbitrary $f(b_j) > 0$, for some unitary V, (3.16) is minimized by

$$S_0 = V\Lambda^{-1/2}U^H \tag{3.18}$$

and (3.16) becomes

$$J(S_0) = \sum_{i=0}^{M-1} f(b_i)a_i^2 \tag{3.19}$$

with a_i defined as $\left[V\Lambda V^H\right]_{ii}$

Before proceeding, we need some results from the theory of majorization, [3]. The first is the definition of majorization

Definition 3.1 *Consider two sequences $x = \{x_i\}_{i=1}^n$ and $y = \{y_i\}_{i=1}^n$ with $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$. Then we say that y majorizes x, denoted as $x \prec y$, if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ holds for $1 \leq k \leq n$, with equality at $k = n$.*

The second is the definition of Schur-Concave function.

Definition 3.2 *A real valued function $\phi(z) = \phi(z_1, \ldots, z_n)$ defined on a set $\mathcal{A} \subset R^n$ is said to be Schur concave on $\mathcal{A}$ if $x \prec y$ on $\mathcal{A} \Rightarrow \phi(x) \geq \phi(y)$. ϕ is strictly Schur concave on $\mathcal{A}$ if strict inequality $\phi(x) > \phi(y)$ holds when x is not a permutation of y.*

The following Lemma provides a test for Schur-Concavity, [3].

Lemma 3.1 *Let $g(z)$ be a scalar real valued function defined and continuous on $\mathcal{D} = \{(z_1, \ldots, z_n) : z_1 \geq \ldots \geq z_n\}$, and twice differentiable on the interior of $\mathcal{D}$. Then $g(z)$ is Schur concave on $\mathcal{D}$ if $\frac{\partial g(z)}{\partial z_k}$ is increasing in k.*

Then the following obtains from [6]. If under optimum bitloading, the function in (3.19) is Schur-Concave in a_i, then for a suitable permutation matrix P, and U as in (3.17), the optimizing S_0 can be chosen as

$$S_0 = PU^H.$$

Thus because of (2.10), both G_0 and S_0 can be chosen as Orthonormal. Further, in this case the optimizing A obeys

$$A = -U_0^H R_u U_1 \left(U_1^H R_u U_1\right)^{-1} \tag{3.20}$$

which is independent of S_0 and the optimum bit rate allocation b_i, just as the optimizing S_0 is also independent of b_i, i.e. the separation principle described earlier holds.

4. THE MAIN RESULT

In view of the above we need to find conditions on $f(.)$ that ensure that the function in (3.19) is Schur-Concave in a_i. We now prove that this requires three conditions on $f(.)$.

Condition A: For all $b > 0$, (i) $f(b)$ is positive monotonic increasing function, (ii) $f(b)$ is a convex function, (iii) $\log f(b)$ is a concave function for $b \geq 1$, and (iv) f(0)=0.

Since under (i), $f(b)a_i^2$ is also convex, [2] shows that under (2.2), for a given set of a_i, the following non-negative integer bit-loading algorithm minimizes (3.19). Define

$$\delta_i(b) = (f(b) - f(b-1))a_i^2,$$

and let S denote the set of smallest B elements of

$$\tau = \{\delta_k(x) : k = 1, \ldots, N, x = 1, \ldots, B\}$$

Then optimal solution $b^* = [b_1^*, \ldots, b_N^*]^T$ is defined as

$$b_k^* = \begin{cases} 0 & : \quad \delta_k(1) \notin S \\ B & : \quad \delta_k(B) \in S \\ y & : \quad \delta_k(y) \in S, \delta_k(y+1) \notin S \end{cases} \tag{4.21}$$

Notice in particular under optimum bit loading for all positive j, l

$$[f(b_j) - f(b_j - 1)]a_j^2 < [f(b_l + 1) - f(b_l)]a_l^2. \tag{4.22}$$

The following lemma is useful.

Lemma 4.1 *Under (i), (iii) and (iv) of Condition A, if $a_1 > a_2 > 0$, $0 < b_1 < b_2$ and $f(b_1)a_1 > f(b_2)a_2$, then $[f(b_1) - f(b_1 - 1)]a_1 > [f(b_2) - f(b_2 - 1)]a_2$.*

Proof: If $b_1 = 1$, then the result holds because of (iv). If $b_1 > 1$, as $\log f(.)$ is concave, $\log(f(b_1)) - \log(f(b_1 - 1)) > \log(f(b_2)) - \log(f(b_2 - 1))$. This in turn implies that

$$a_1/a_2 < f(b_1)/f(b_2) < (f(b_1) - f(b_1 - 1))/(f(b_2) - f(b_2 - 1)).$$

■

Then the main result follows.

Theorem 4.1 *Suppose in (3.19) $a_i > 0$. Under optimum nonnegative integer bit allocation $J = \sum_{i=0}^{M-1} f(b_i)a_i^2$ is Schur concave in a_i if Condition A holds.*

Proof: Suppose $a_j > a_l > 0$, a necessary condition for J to be minimum is $0 \leq b_j < b_l$, [3].

Now suppose that J is not Schur concave. Then from Lemma 3.1, if $a_j > a_l$ then one also has $\frac{f(b_j)}{f(b_l)} > \frac{a_l}{a_j}$. If $b_j = 0$ then this cannot hold. Now suppose $b_j > 0$. Observe that (4.22) holds. From condition (iii) we have $\frac{f(b_j+1)}{f(b_j)} > \frac{f(b_l+1)}{f(b_l)}$, thus $\frac{f(b_j+1)}{f(b_l+1)} > \frac{a_l}{a_j}$. Applying Lemma 4.1,

$$[f(b_j) - f(b_j - 1)]a_j > [f(b_l) - f(b_l - 1)]a_l \tag{4.23}$$

$$[f(b_j + 1) - f(b_j)]a_j > [f(b_l + 1) - f(b_l)]a_l \tag{4.24}$$

From condition (i), combine (4.22) and (4.23) gives

$$\frac{f(b_l) - f(b_l - 1)}{f(b_l + 1) - f(b_l)} < \frac{a_l}{a_j} \tag{4.25}$$

From $b_j < b_l$, we have $b_j + 1 \leq b_l$, and $f(b_j + 1) - f(b_j) \leq f(b_l) - f(b_l - 1)$ from condition (ii), combine this with (4.24) gives

$$\frac{f(b_l) - f(b_l - 1)}{f(b_l + 1) - f(b_l)} \geq \frac{a_l}{a_j} \tag{4.26}$$

which contradicts (4.25), and completes the proof. ■

5. EXAMPLE: PAM

Next we give an example of $f(b)$ satisfying the conditions in Theorem 4.1. Suppose the modulation scheme in each subchannel is PAM. Then the SNR required to achieve a SER η is given by [9]

$$\eta = 2(1-2^{-b})Q\left(\sqrt{\frac{3\text{SNR}}{(2^b-1)}}\right) \tag{5.27}$$

where

$$Q(a) = \int_a^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \tag{5.28}$$

from which we obtain SNR $f(b)$ as

$$f(b) = \left[Q^{-1}\left(\frac{\eta}{2(1-2^{-b})}\right)\right]^2 \left(\frac{2^{2b}-1}{3}\right) \tag{5.29}$$

This holds for $b > -\log_2(1-\eta)$. We now make the reasonable assumption that the target $\eta < 0.5$. Then (5.29) is defined for all positive integer b. Further it is reasonable to define $f(0) = 0$, as zero bit transmission requires vanishingly small transmission power for arbitrarily small SER.

Now a variation of a proof in [12] shows that $f(.)$ satisfies (i) and (ii). By definition it satisfies (iv). We focus on proving (iii), by showing that $(\log f(b))'' \leq 0$.

$\log(\frac{2^{2b}-1}{3})$ is concave, since

$$\left(\log\left(\frac{2^{2b}-1}{3}\right)\right)'' = \left(\frac{6(\log 2)2^{2b}}{2^{2b}-1}\right)' = \frac{-6(\log 2)2^{2b}}{(2^{2b}-1)^2} < 0 \tag{5.30}$$

then it remains to prove $\log Q^{-1}\left(\frac{\eta}{2(1-2^{-b})}\right)$ is concave,

Note that $(Q^{-1}(x))' = -\sqrt{2\pi}e^{[Q^{-1}(x)]^2/2}$, and use $Q^{-1}(.)$ and $e^{[Q^{-1}(.)]^2/2}$ in the following for simplicity. First,

$$\left(\log Q^{-1}(.)\right)' = \frac{\sqrt{2\pi}\eta \log 2}{4} \frac{e^{[Q^{-1}(.)]^2/2}(2^{-b})}{Q^{-1}(.)(1-2^{-b})^2} \tag{5.31}$$

then

$$\left(\log Q^{-1}(.)\right)'' = aZ(b) \tag{5.32}$$

where

$$a = \frac{e^{[Q^{-1}(.)]^2/2}\sqrt{2\pi}\eta \log 4(2^{-b})}{8\left[Q^{-1}(.)(1-2^{-b})^2\right]^2} > 0 \tag{5.33}$$

$$\begin{aligned} Z(b) = {} & \sqrt{2\pi}\eta e^{[Q^{-1}(.)]^2/2}(2^{-b})[Q^{-2}(.)-1] \\ & -2Q^{-1}(.)(1-2^{-2b}) \end{aligned} \tag{5.34}$$

Note that $Z(b)$ is a decreasing function of b, then

$$Z(b) \leq Z(-\log_2(1-\eta)) = -\sqrt{2\pi}\eta(1-\eta) \leq 0 \tag{5.35}$$

Hence the proof is completed.

An interesting observation is that with high bit rate assumption, $f(b)$ in (5.29) is approximated as $\frac{1}{3}\left[Q^{-1}(\frac{\eta}{2})\right]^2 2^{2b}$, which can be easily verified to satisfy (i-iii) of Condition A.

6. CONCLUSIONS

In this paper we have given a sufficient condition on modulation schemes that ensure that the optimum DMT under integer bitloading and without a high bit rate assumption, is orthonormal. This same condition ensures a separation principle which has the attractive property that should the channel/interference remain invariant after the initial connection is established, then only bit loading selection needs to be updated in response to the changing service characteristics, i.e. the optimum transceiver itself depends only on the channel/interference conditions and not on the QoS requirements. The condition derived is satisfied by PAM, and we suspect most other existing modulation schemes in an AWGN environment.

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