MINIMUM RATES SCHEDULING FOR OFDM BROADCAST CHANNELS

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ABSTRACT

In this paper, we solve the weighted sum rate problem for an orthogonal frequency division multiplexing (OFDM) broadcast channel (BC) under a sum power constraint, if minimum rates have to be guaranteed in each fading state and perfect channel state information (CSI) is assumed at the base station and the mobiles. The problem is subdivided into two problems. First, we tackle the problem of feasibility, which occurs since the system is power limited and not all required rates might be supportable. Subsequently, the optimal resource allocation in case of feasibility is derived. Moreover, the optimal decoding order, which is not determined by the problem formulation itself, is obtained as the ordering of the Lagrangian factors of the main problem. Finally, we show that the problem is closely related to the weighted rate sum maximization and sum power minimization and all three can be interpreted in a unifying framework embedded in a higher dimensionality.

1. INTRODUCTION

For a major part of future wireless communication systems OFDM is going to be a key technology due to its advantages over CDMA such as frequency selective resource allocation, robustness against intersymbol interference and extremely simple signal processing due to (I)FFT. Not only promising standards already fixed (such as the 802.11 family and 802.16) are based on OFDM, but also OFDM will likely become a predominant core technology for future 3G+ and 4G mobile communication systems. In this context, optimal resource allocation is a key challenge due to the scarcity and expense of bandwidth and power. To derive efficient algorithms for real world systems, the understanding of the underlying information theoretic problems is of great importance. Not only it provides a fundamental upper bound for all feasible schemes due to the idealized assumptions (such as e.g. Gaussian codebooks and ideal successive interference cancellation (SIC)), but also gives insights into the problem structure. Using recently established duality of BC and multiple access channel (MAC) [1], all results can be carried over from uplink to downlink and vice versa. Furthermore these theoretical results can be used to describe quite well real world systems by using the SNR gap approximation.

In a cellular OFDM system, the base station has to be aware of different constraints. First, it has to keep the buffer queues finite in order to avoid overflows and hence to stabilize the system. In each time slot a certain amount of power is available, since inter cell interference has to be limited and peak power constraints might be imposed by the used amplifiers. This leads to an optimization problem, which maximizes a queue length-weighted sum of rates [2].This is a well known approach and was studied for the (MIMO-) OFDM downlink in [3, 4]. We can interpret this point of view as a fixed power strategy, where unfortunately fairness can not be guaranteed and the near-far problem might lead to unacceptable long delays. A second type of constraints the base station has to take into account are rate requirements originating from Quality of Service (QoS) demands (e.g. in form of delay constraints). This perspective leads to a sum power minimization problem for given rate requirements gaining interest recently. It was analyzed for a fixed decoding order in for OFDM [5] and multiple input multiple output (MIMO) [6] channels leading to nonconvex formulations. The general OFDM case was solved recently in [7].

In this paper we combine the two perspectives to a more advanced and more realistic problem, where the limited sum power has to be allocated optimally such that the system is stabilized and certain minimum rates are guaranteed in each fading state. This is a perspective similar to [8], where the problem of guaranteeing rates in each fading state for a flat scalar BC was addressed from an ergodic viewpoint. However, this approach can not be carried over to OFDM. We show how our problem is connected to the sum power minimization and the weighted sum rate maximization. Subsequently, we give a condition for feasibility and derive the optimal decoding order as a part of the solution. Then we present an algorithm yielding the optimal resource allocation and give a Lagrangian interpretation of the enhanced joint set of rates and sum power $(\mathbf{R}, P)$.

The remainder of this paper is organized as follows. Section 2 contains the system model and the problem statement. In Section 3, a unifying perspective is presented and the solution to the problem is derived. Subsequently, an efficient algorithm is presented. Finally we conclude with some remarks in Section 4.

2. SYSTEM MODEL AND PROBLEM STATEMENT

Supposing familiarity with OFDM we assume an OFDM BC with K subcarriers, M users and a short term sum power constraint $\sum_{m,k=1}^{M,K} \mathbb{E}\{|x_{m,k}|^2\} \leq \bar{P}$, where $x_{m,k}$ is the signal transmitted to user m on subcarrier k and $\mathbb{E}\{.\}$ stands for the expectation operator. Then the system equation on each subcarrier can be written as

$$y_{m,k} = h_{m,k} \sum_{j \in \mathcal{M}} x_{j,k} + n_{m,k} \quad \forall k \tag{1}$$

where $y_{m,k}$ is the signal received by user m on subcarrier k, $n_{m,k} \sim \mathcal{CN}(0, \sigma^2)$ is circular symmetric additive white Gaussian noise with variance σ^2 and $\mathcal{M} = \{1, ..., M\}$ denotes the set of users.

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Let $\mathbf{h} = [h_{1,1}, \ldots, h_{1,K}, h_{2,1}, \ldots, ...h_{M,K}]^T$ denote the stacked vector of channel coefficients. We assume that superposition coding is performed at the base station having full non-causal knowledge of all messages to be transmitted. The users decode their messages using successive interference cancellation. Let $\pi \in \Pi$ be an arbitrary decoding order from the set of all $M!$ possible decoding orders, such that user $\pi(M)$ is decoded first, followed by user $\pi(M-1)$ and so on. Then the rate of user $\pi(m)$ can be expressed as

$$\tilde{R}_{\pi(m)} = \sum_{k=1}^{K} \log\left(1 + \frac{|h_{\pi(m),k}|^2 p_{\pi(m),k}}{\sigma^2 + |h_{\pi(m),k}|^2 \sum_{n<m} p_{\pi(n),k}}\right), \quad (2)$$

where all logarithms are to the base e in the following. The capacity region of the OFDM BC under a given sum power constraint $\bar{P}$ is given by

$$\mathcal{C}_{BC}(\mathbf{h}, \bar{P}) \equiv \bigcup_{\substack{\pi\in\Pi \\ \sum_{m,k=1}^{M,K} p_{m,k} \leq \bar{P}}} \left\{\mathbf{R} : R_{\pi(m)} \leq \tilde{R}_{\pi(m)}\ , m \in \mathcal{M}\right\}$$

where $\tilde{R}_{\pi(m)}$ is defined in equation (2) and $\mathbf{R} = [R_1, ..., R_M]^T$ denotes the vector of rates. Now let

$$v_k = \sum_{j=1}^{M} h_{m,k} u_{m,k} + n_k \quad \forall k \quad (3)$$

be the system equation for the dual MAC with v_k being the received signal on carrier k, $u_{m,k}$ the signal transmitted from user m on carrier k and $n_k \sim \mathcal{CN}(0, \sigma^2)$. We know from recent results [1, 9, 10], that the capacity regions OFDM MAC and BC are dual:

$$\mathcal{C}_{BC}(\mathbf{h}, \bar{P}) \equiv \mathcal{C}_{MAC}(\mathbf{h}, \bar{P}) \quad (4)$$

This allows to solve problems for the broadcast channel in the dual multiple access channel (MAC) and to relate them to the BC by uplink-downlink duality and vice versa. For the ease of notation, we omit the subscript $\cdot_{BC}$ in the following.

2.1. Problem Statement

In this paper, we are interested in the solution to the following problem:

Problem 1 [Main Problem]

$$\begin{aligned} &\textit{maximize} \quad \sum_{m=1}^{M} \mu_m R_m \\ &\textit{subject to} \quad R_i \geq \bar{R}_i \quad \forall i \in \mathcal{S}, \mathcal{S} \subseteq \mathcal{M} \\ &\qquad\qquad\quad \mathbf{R} \in \mathcal{C}(\mathbf{h}, \bar{P}) \end{aligned} \quad (5)$$

The first condition reflects the minimum rate requirements for all users in $\mathcal{S}$ and the second condition is due to the sum power constraint $\bar{P}$. To derive the solution to the stated problem, we characterize two subproblems: The first is the well known maximization of a weighted sum of rates for a given channel and under a given sum power constraint $\bar{P}$ [3].

Problem 2 [Weighted Sum Rates Problem]

$$\textit{maximize} \sum_{m=1}^{M} \mu_m R_m \quad \textit{subj. to} \quad \mathbf{R} \in \mathcal{C}(\mathbf{h}, \bar{P}) \quad (6)$$

The second is the recently solved sum power minimization problem [7]:

Problem 3 [Minimum Sum Power Problem]

$$\begin{aligned} \textit{minimize} \quad P \quad \textit{subj. to} \quad & R_i \geq \bar{R}_i \quad \forall i \in \mathcal{S}, \mathcal{S} \subseteq \mathcal{M} \\ & \mathbf{R} \in \mathcal{C}(\mathbf{h}, P) \end{aligned} \quad (7)$$

3. OPTIMUM RESOURCE ALLOCATION

3.1. A global perspective

We begin by deriving a unique framework for all problems in a higher dimensionality. By the concavity of the log-function it is easy to prove the following lemma:

Lemma 1 *Define the set* $\mathcal{G}(\mathbf{h}) = \{\mathbf{R}, P : \mathbf{R} \in \mathcal{C}(\mathbf{h}, P)\}$. *The set* $\mathcal{G}(\mathbf{h})$ *is a convex set.*

Obviously, the duality of MAC and BC holds for the enhanced set $\mathcal{G}(\mathbf{h})$, too. By Lemma 1 there are $\lambda \in \mathbb{R}_{++}$ and $\tilde{\mu} \in \mathbb{R}_{+}^{M}$ such that Problems 1-3 are equivalent to solving the following problem:

$$\max \sum_{m=1}^{M} \tilde{\mu}_m R_m - \lambda P \ \text{ subj. to } \ (\mathbf{R}, P) \in \mathcal{G}(\mathbf{h}) \quad (8)$$

This is a convex problem, since the function in (8) is affine and the set $\mathcal{G}(\mathbf{h})$ is a convex set. To see the coherence of all problems, note that in Problem 2 the Lagrangian factors $\tilde{\mu}$ are known and λ is the power price. In Problem 3 we have $\lambda = 1$ and the vector $\tilde{\mu}$ constitutes the Lagrangian vector of the rate requirements. The relation to Problem 1 will be pointed out in the remainder of this paper. In all cases we search for parts of the Lagrangian normal vector to the supporting hyperplane and parts of the optimal supported point on the boundary of $\mathcal{G}(\mathbf{h})$. We will see further, that in combination with the subcarrier-wise rate-power relation a general tool for designing iterative algorithms is available. In this case the $\tilde{\mu}$ have the additional interpretation as water-filling levels.

3.2. Maximization of a weighted sum of rates

In [3] it was shown via uplink-downlink duality and by exploiting the properties of polymatroids, that Problem 2 can be reformulated as a convex optimization problem. In [10], Tse presented an algorithm for solving this problem for parallel Gaussian channels directly in the downlink. This algorithm can be carried over to the OFDM BC case. He introduces the elegant notion of *marginal utility functions* to characterize the revenue of each user to the objective function:

$$u_m^{(k)}(z) = \frac{\mu_m}{\left(\sigma_{m,k}^2 / |h_{m,k}|^2 + z\right)} - \lambda \quad (9)$$

The set of equations characterizing the solution is given by

$$R_m = \int_0^\infty \sum_{k: u_m^{(k)}(z) = \left[\max_i u_i^{(k)}(z)\right]^+} \frac{1}{\left(\sigma_{m,k}^2 / |h_{m,k}|^2 + z\right)} dz \quad (10)$$

$$P(\lambda) = \sum_{k=1}^{K} \left[\max_m \left(\frac{\mu_m}{\lambda} - \sigma_{m,k}^2 / |h_{m,k}|^2\right)\right]^+ \quad (11)$$

where λ is the Lagrangian parameter of the sum power constraint and $[a]^+ := \max(0, a)$. For a detailed study the reader is referred to [10, 11]. The solution can be found by solving equation (11) for the Lagrangian multiplier λ. This can be done by simple bisection, since the RHS of (11) is monotone in λ.

Algorithm 1 Weighted rate sum maximization

(1) solve (11) for Lagrangian factor λ
(2) determine intersections of marginal utility functions (9) for all K
(3) calculate resulting rates (10)

3.3. Sum power minimization

Problem 3 was solved in [7] recently. The problem is considered in the dual MAC, and virtual subcarrier specific decoding orders $\pi_k \in \Pi \quad \forall k$ are introduced such that

$$|h_{\pi_k(1),k}|^2 \geq |h_{\pi_k(2),k}|^2 \geq \cdots \geq |h_{\pi_k(M),k}|^2. \qquad (12)$$

We call these virtual, since they are only a mathematical tool to obtain the solution including the global decoding order. In principle, the OFDM MAC is a *nondegraded* channel. Nevertheless, it can be decomposed to a union over *degraded* channels. This allows to exploit the known properties of degraded channels such as the optimal decoding order. This leads to a log-convex formulation of the problem. Having in mind the Lagrangian interpretation, we note the following: to determine the optimal decoding order the resulting Lagrangian multipliers $\{\mu_m\}_{m=1}^M$ have to be ordered, since they determine the optimal vertex of the corresponding polymatroid. Interestingly, as shown in [7], the problem is not convex for fixed decoding orders - not even for the optimal decoding order. Define coefficients $n_{m,k}$ as

$$n_{m,k} := \log \left\{ e^{\sum\limits_{n>m} R_{\pi_k(n),k}} \left[\frac{\sigma^2}{|h_{\pi_k(m),k}|^2} + \sum_{j=1}^{m-1} \frac{\sigma^2}{|h_{\pi_k(j),k}|^2} \left(e^{R_{\pi_k(j),k}} - 1 \right) e^{\sum\limits_{n=j+1}^{m-1} R_{\pi_k(n),k}} \right] \right\}^{-1}. \qquad (13)$$

Then the Karush-Kuhn-Tucker (KKT) conditions can be written in a form allowing water-filling:

$$R_{\pi_k(m),k} = \left[\log(\mu_m) + n_{m,k} \right]^+, \quad \forall m, k \qquad (14)$$

This motivates Algorithm 2, which shows excellent convergence behavior due to the log-convexity and the semi-analytic stepwise solution through water-filling.

Algorithm 2 Iterative "rate water-filling"

set $R_{m,k} = 0 \quad \forall m \in \mathcal{M},\ k = 1, ..., K$
while desired accuracy is not reached **do**
  for $m = 1$ to M **do**
    (1) compute the coefficients $n_{m,k}$ (13) for user m
    (2) do water-filling with respect to the rates $R_{m,k}$ for user m as in equation (14)
  end for
end while

Theorem 1 (proven in [7]) *Algorithm 2 converges to a stationary point P_{min}, which is the global optimum of Problem 3.*

Note, that in principle the formulation from [10] can be used to solve the problem; however, it is much more complicated, since in each step varying λ all intersections of all relevant marginal utility functions have to be calculated on each carrier.

3.4. The main problem

With the insights of the preceding sections, we turn to Problem 1: In contrast to the two previously considered problems, which are always feasible, the feasibility of Problem 1 is not guaranteed. The solution to the sum power minimization problem characterizes feasibility of the main problem *implicitly*. Obviously the main problem is feasible, if the minimum power to fulfill the vector of rate constraints P_{min} smaller than the power budget $\bar{P}$. Otherwise, the feasible set of Problem 1 is empty. Let us first interpret the solution in terms of (8).

3.4.1. Lagrangian interpretation of the solution

Define the vector $\tilde{\mu} = [\mu_1 + \mu_1^*, ..., \mu_M + \mu_M^*]^T$. The components μ_m are the weight factors from (5) and the components μ_m^* are the Lagrangian multipliers of the rate constraints in (5). Further λ is the Lagrangian multiplier due to the power constraint $\bar{P}$. Then the vector $[\tilde{\mu} \ \ \lambda]^T$ is the normal vector of the supporting hyperplane to the region $\mathcal{G}(\mathbf{h})$ and the supported point is the solution $(\mathbf{R}^*, \bar{P})$ to (5). The Lagrangian factor μ_m^* is strictly greater than zero if and only if the rate constraint of user m is active. Then μ_m^* delivers a revenue to the corresponding normal vector of the supporting hyperplane, assuring the minimum rate.

3.4.2. Optimal decoding order

To derive the optimal decoding order, we focus on the Lagrangian interpretation: we know that the vector $\mathbf{R}^*$ is the solution to the optimization problem:

$$\max \sum_{m=1}^{M} \tilde{\mu}_m R_m \ \text{ subj. to } \ \mathbf{R} \in \mathcal{C}(\mathbf{h}, \bar{P}) \qquad (15)$$

The point is achieved with an decoding order π^* such that

$$\tilde{\mu}_{\pi^*(M)} \geq \tilde{\mu}_{\pi^*(M-1)} \geq \cdots \geq \tilde{\mu}_{\pi^*(1)} \qquad (16)$$

and the solution $\mathbf{R}^*$ is the vertex $\mathbf{R}^{\pi^*}$ of the corresponding polymatroid. Note, that the optimal decoding order can *not* be determined a priori, since the required weight vector $\tilde{\mu}$ - including the Lagrangian factors of the rate requirements - is not known. The weights μ constitute only a part of the overall normal vector. Hence, the optimal decoding order is part of the solution. This interesting fact is in analogy to [7], where the Lagrangian weight vector μ is completely unknown a priori.

3.4.3. Algorithmic Solution

To solve Problem 1, we combine the two presented algorithms: In a first step, feasibility is tested by sum power minimization. In case of feasibility, in each step the Lagrangian parameters μ_m are adjusted for all users with violated rate constraints.

Algorithm 3 Minimum Rates Algorithm

(1) check feasibility with Algorithm 2
while desired accuracy not reached **do**
  for $m = 1$ to M **do**
    if $R_m < \bar{R}_m$ **then**
      (2) adjust μ_m such that $R_m = \bar{R}_m$ using Algorithm 1
    end if
  end for
end while

Alternatively, we propose the following algorithm which is based on eq. (8). Depending on whether the rate constraints are active, the water-filling level is adjusted such that the requirements are fulfilled. We have the following theorem ensuring convergence:

Algorithm 4 Alternative Minimum Rates Algorithm

(1) check feasibility with Algorithm 2
(2) choose initial Lagrangian factors λ_+ and λ_-
while sum power constraint $\bar{P}$ is not met **do**
  while desired accuracy not reached **do**
    for $m = 1$ to M **do**
      (3) compute the coefficients $n_{m,k}$ (13) for user m
      (4) do water-filling with fixed level μ_m (14)
      if $R_m < \bar{R}_m$ **then**
        (5) choose water-filling level $\log(\mu_m + \mu_m^*)$ such that $R_m = \bar{R}_m$
      end if
    end for
  end while
  (6) increase (decrease) λ if $P > \bar{P}$ ($P < \bar{P}$) by bisection
end while

Theorem 2 (from [12]) *Algorithm 3 and 4 converge to a stationary point R^*, which is the global optimum of Problem 1.*

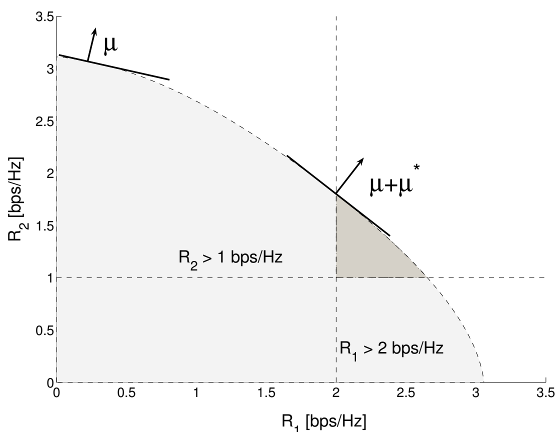

Fig. 1. Example for rate requirements $R_1 \geq 2$ and $R_1 \geq 1$ and $\mathcal{C}(\mathbf{h}, \bar{P})$ for 256 subcarriers at $10dB$ with original weight vector $\boldsymbol{\mu}$ and resulting Lagrangian vector $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} + \boldsymbol{\mu}^*$

3.5. A unifying framework

Interestingly, all problems presented in this paper can be solved with algorithms based on the KKT-conditions given in equation (14). This is possible since a unique bijective transformation between the rates R_m and the corresponding powers p_m exists in the case of SISO BC and MAC in general. Fortunately this also holds per subcarrier for OFDM . Note, that it does not apply to the MIMO case. The set of equations in (14) combined with the Lagrangian interpretation on the set $\mathcal{G}(\mathbf{h})$ allows an iterative optimization also for the problem of maximizing a weighted sum of rates. So iterative rate water-filling for *fixed levels* (fixed Lagrangian multipliers $\boldsymbol{\mu}$), delivers a convergent algorithm for the maximization of a weighted sum of rates. Nevertheless, Algorithm 2 is more efficient, since it solves for the rates only once, after the correct Lagrangian multiplier λ was determined.

4. CONCLUSIONS

In this paper we characterized the solution to the problem of maximizing the stability region while guaranteeing minimum rates for the OFDM broadcast channel. We showed that this is a convex optimization problem and gave a condition for feasibility. Further, we showed that the problem can be interpreted in a higher dimension in terms of the enhanced convex set $(\mathbf{R}, P)$ introduced in [11]. It was shown, that the sum power minimization problem from [7] and the weighted sum maximization problem are related to the considered problem in $(\mathbf{R}, P)$ and all three problems can be seen as the optimization of a linear function over a convex set. We then presented an efficient algorithm, which solves the problem and proved convergence. Further, we derived the optimal decoding order, which can be determined from the users weights and the Lagrangian factors for the rate requirements, if they are active. Due to duality, all results hold for the uplink as well.

5. REFERENCES

[1] N. Jindal, S. Vishwanath, and A. Goldsmith, "On the duality of Gaussian multiple-access and broadcast channels," *IEEE Trans. Inform. Theory*, vol. 50, no. 5, pp. 768–783, May 2004.

[2] E. Yeh and A. Cohen, "Throughput and delay optimal resource allocation in multiaccess fading channels," in *Proc. IEEE Int. Symp. Information Theory (ISIT)*, Yokohama, 2003, p. 245.

[3] G. Wunder and T. Michel, "On the OFDM multiuser downlink capacity," in *Proc. IEEE Vehicular Techn. Conf. (VTC)*, Los Angeles, Sep 2004.

[4] T. Michel and G. Wunder, "Optimal and low complexity suboptimal transmission schemes for MIMO-OFDM broadcast channels," in *Proc. IEEE Int. Conf. on Communications (ICC)*, May 2005.

[5] J. Oh, S.-J. Kim, and J. Cioffi, "Optimum power allocation and control for OFDM in multiple access channels," in *Proc. IEEE Vehicular Techn. Conf. (VTC)*, Los Angeles, Sep 2004.

[6] J. Oh, S.-J. Kim, R. Narasimhan, and J. Cioffi, "Transmit power optimization for Gaussian vector broadcast channels," in *Proc. IEEE Int. Conf. on Communications (ICC)*, Seoul, May 2005.

[7] T. Michel and G. Wunder, "Solution to the sum power minimization problem under given rate requirements for the OFDM multiple access channel," in *Proc. Annual Allerton Conf. on Commun., Control and Computing, Monticello, USA*, 2005.

[8] N. Jindal and A. Goldsmith, "Capacity and optimal power allocation for fading broadcast channels with minimum rates," *IEEE Trans. Inform. Theory*, vol. 49, no. 11, pp. 2895–2909, Nov 2003.

[9] S. Vishwanath, N. Jindal, and A.J. Goldsmith, "On the capacity of multiple input multiple output broadcast channels," in *Proc. IEEE Int. Conf. on Communications (ICC)*, New York, April 2002.

[10] D. Tse, "Optimal power allocation over parallel Gaussian broadcast channels," unpublished, 1998.

[11] D.N.C. Tse and S.V. Hanly, "Multiaccess fading channels - part I: Polymatroid structure, optimal resource allocation and throughput capacities," *IEEE Trans. Inform. Theory*, vol. 44, no. 7, pp. 2796–2815, Nov 1998.

[12] G. Wunder and T. Michel, "Optimal resource allocation for OFDM multiuser systems," submitted.