MARKOV MODELS FOR LIMITED FEEDBACK MIMO SYSTEMS

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ABSTRACT

Multiple antenna wireless systems with feedback of quantized channel information, called "limited feedback" systems, are attractive choices for improving the quality of downlink (DL) transmission. Most work in this area use the block-fading channel model where the DL channel is assumed constant in each block and different blocks uncorrelated. In this paper, we consider limited feedback for a temporally correlated DL channel. Markov models are introduced for characterizing the temporal correlation and probability distribution of the DL channel. Using the Markov models, average feedback rates are derived. Numerical results show that the feedback rates are proportional to the Doppler frequency.

1. INTRODUCTION

For multiple-input-multiple-output (MIMO) systems, perfect channel state information (CSI) at the transmitter can be used to maximize data rates or link reliability against fluctuation of wireless channels. Sending back the MIMO channel information may require a high-rate channel due to the multiplicity of the channel coefficients. Limited feedback provides practical solutions for this problem. The idea of limited feedback is to use a very low-rate feedback channel to send back CSI while ensuring near-optimal performance.

In this paper, we consider the scenario where there exist a downlink (DL) MIMO system that transmits data from a base-station to a mobile terminal, and a limited feedback system that transmits the DL CSI in the reverse direction using a finite-rate feedback channel. At the mobile terminal, the DL CSI is quantized into a finite set of channel states and the index of the current channel state is transmitted to the base station. Based on the index, the base station selects an appropriate transmission strategy such as a precoding matrix or a beamforming vector.

The assumption of *block fading* for the DL channel is made in majority of the papers in the area of limited feedback (see [1] and references therein). As the result, different DL channel realizations are assumed independent and hence not correlated in time [2]. The block fading assumption allows the design of a limited feedback system to be formulated as a line-packing problem [3, 4] or a vector quantization problem [5,6]. However, the main drawback of the above assumption is that how much and how frequent the feedback of CSI should be can not be accurately analyzed because both of them depend on the temporal correlation of the DL channel. This fact motivates the development of efficient feedback algorithms in [7, 8], which exploit channel temporal correlation in . Nevertheless, there exists no study of the feedback rate for a temporally correlated (TC) channel in the literature. That provides motivation for the investigation of the following problem in this paper. Answers to this problem will provide useful guidelines for allocating uplink resources such as bandwidth and time slots to the feedback channel.

Problem: *How does the average feedback rate depend on the temporal correlation and distribution of the DL channel?*

The main contributions of this paper are the development of Markov models for the temporally correlated CSI in a MIMO limited feedback system and the use of the Markov models for analyzing the average feedback rate. A quantizer at the mobile terminal partitions the continuous channel space into a finite set of channel states that form the Markov state space while the temporal correlation and probabilistic distribution of the channel are captured by the properties of the Markov models. Using the Markov models, we derive the average feedback rates. We also show that that the average feedback rates are proportional to the Doppler frequency. More details are given as follows.

Two types of limited feedback systems are considered. First we consider a MIMO system, where the "gain" of the channel, defined as the squared Frobenius norm of the channel matrix, constitutes the CSI. This is a representative feedback scenario for multi-user scheduling, adaptive modulation and power control. The second limited feedback system feeds back the dominant channel singular vector. This models a MIMO beamforming system with quantized feedback of the beamforming vector. For each of the above systems, a Markov model of the quantized DL channel is constructed and an expression of the average feedback rate is derived. Furthermore, we demonstrate with numerical results that the average feedback rate is proportional to the Doppler frequency.

2. SYSTEM DESCRIPTION

Consider a MIMO system with M_t transmit antennas at the base-station and M_r receive antennas at the mobile. The narrowband matrix channel at baseband for downlink transmission is represented by a $M_r \times M_t$ complex matrix $\mathbf{H}(t)$. The elements of $\mathbf{H}$ are i. i. d. $\mathcal{CN}(0,1)$ and t is the sampling time instant while the time-variation is modeled by the Clark's correlation function [9]. Without loss of generality, the mobile is assumed to have perfect knowledge of $\mathbf{H}(t)$. It is also assumed that there exists a low-rate, zero-delay, error-free feedback channel for sending quantized channel information from the mobile to the base-station. In particular, we consider two classes of feedback systems described below.

In the first case, we consider feedback of channel gain defined by the Frobenius norm of the channel matrix expressed as

$$g(t) = \|\mathbf{H}(t)\|_F^2. \tag{1}$$

Consider a scalar quantizer with N levels (or Voronoi regions) at the mobile. Based on squared error $g(t)$ is mapped to a particular level given by $\hat{g}(t)$. The goal is to characterize the feedback frequency (or feedback rate) for this scheme as a function of the Doppler frequency. The average feedback frequency is defined as the average number of times feedback is performed per symbol period. The average feedback rate is the product of the average feed-

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back frequency and the number of bits for each feedback, which depends on the feedback strategy.

The second system that is analyzed is a beamforming and maximum ratio combining (MRC) system with quantized feedback [10, 11]. The system equation is given by

$$\mathbf{y}(t) = \mathbf{H}(t)\mathbf{u}(t)x + \mathbf{n}(t) \quad (2)$$

where $\mathbf{u}(t)$ is the beamformer, $\mathbf{y}(t)$ is the received signal vector, x is the complex symbol and $\mathbf{n}(t)$ is the receiver noise having entries that are i. i. d. $\mathcal{CN}(0, 1)$. At the mobile, the beamformer $\mathbf{u}(t)$ is chosen from a set of N predetermined beamformers, called a codebook, such that the instantaneous SNR is maximized. It is assumed that all the N beamformers are adjacent to each other and thus if at time t, the beamformer chosen is different from that at time $t-1$, a $\log_2 N$ bit feedback is generated, otherwise no feedback is needed. The aim here is to characterize the average feedback frequency (or alternatively rate) as a function of the Doppler frequency.

3. FEEDBACK OF QUANTIZED CHANNEL GAIN

In this section, we consider a MIMO system having feedback of quantized channel gain. The temporally correlated quantized channel gain is modeled using a Markov chain, referred to as the *gain Markov model*. The state-space of the Markov model is defined by a codebook described in Section 3.1.1 and its properties analyzed in Section 3.1.2 and 3.1.3. Considering a 1-bit feedback, the feedback frequency and hence the feedback bit rate can be derived using the gain Markov model as shown in Section 3.2. The time index of the gain g and the beamformer $\mathbf{u}$ is dropped henceforth.

For the gain Markov model, we make the following assumption. This assumption allows us to use the results in [12] for deriving properties of the gain Markov model. Nevertheless, the proposed Markov model can be readily modified for other types of channel correlation functions.

AS1) *The elements of $\mathbf{H}$ have the Clark's correlation function [9]. Hence, for $1 \leq i \leq M_r$ and $1 \leq j \leq M_t$,*

$$E[h^*_{i,j}(t)h_{i,j}(t + \tau)] = J_0(2\pi f_D\tau) \quad (3)$$

where J_0 is Bessel function of 0th order, and f_D the Doppler frequency.

3.1. Gain Markov Model

It is easy to see that the gain defined in (1) follows a chi-squared distribution with degree of freedom equal to M_rM_t. Hence the gain Markov model is a generalization of the Markov chain in [13] that model the SISO Rayleigh channel. The gain Markov model has N_g states that are defined by the N_g entries of the gain codebook as discussed in the following.

3.1.1. State Space of the Gain Markov Model

The finite states of the gain Markov model form a codebook and we name it as the *gain codebook*. We design the gain codebook, denoted as $\mathbb{G}$, using the Lloyd algorithm [14] such that the average of the following distortion function is minimized:

$$d_g(g; \hat{g}) = (g - \hat{g})^2 \quad (4)$$

where $\hat{g}$ represents the quantized channel gain. The resultant gain codebook is comprised of N_g positive values: $\hat{g}_1, \cdots, \hat{g}_{N_g}$. Each of them is mapped to a region (Voronoi cell) on the positive real line: $\mathbb{R}^+$. For convenience, we define the boundary between the ith and $i+1$ Voronoi cells as

$$\hat{\eta}_i = (\hat{g}_i + \hat{g}_{i+1})/2 \quad (5)$$

with $\hat{\eta}_0 = 0$ and $\hat{\eta}_{N_1+1} = \infty$. Let $\mathcal{X}_i$ be the ith Voronoi cell and it can be written using (5) as

$$\mathcal{X}_i = \left\{\eta \in \mathbb{R}^+ : \hat{\eta}_{i-1} \leq g \leq \hat{\eta}_i\right\}. \quad (6)$$

Thus, each Markov state has a one-to-one correspondence with a particular Voronoi cell of the quantizer. The steady and transition probabilities of the gain Markov model are derived in Section 3.1.2 and 3.1.3, respectively.

3.1.2. Steady-State Probabilities

The steady-state probability of the ith state of the gain Markov model can be expressed as

$$P^{(g)}_i = \Pr(g \in \mathcal{X}_i) = \int_{\hat{\eta}_{i-1}}^{\hat{\eta}_i} f_g(g)dg \quad (7)$$

where $f_g(g)$ is the PDF of g and hence the chi-squared distribution function [15]. From (7), we can show that

$$P^{(g)}_i = \frac{\Gamma(L, \hat{\eta}_{i-1}) - \Gamma(L, \hat{\eta}_i)}{(L-1)!}, \quad (8)$$

where $L = 2M_rM_t$ and $\Gamma(L, x)$ is the incomplete Gamma function defined as [16]

$$\Gamma(L, x) = \int_x^\infty \tau^{L-1} \exp(-\tau)d\tau. \quad (9)$$

For L being a integer, the above incomplete Gamma function has a closed-form expression [16]. As the resultant, the steady-state probability in (8) can be written as: $1 < i < N_2$

$$P^{(g)}_i = \sum_{k=0}^{L-1} \frac{\exp(-\hat{\eta}_{i-1})\hat{\eta}^k_{i-1} - \exp(-\hat{\eta}_i)\hat{\eta}^k_i}{k!}. \quad (10)$$

3.1.3. Transition Probabilities

Limited feedback is designed usually for very slow-fading channel as otherwise the feedback frequency is too high to be practical. Therefore, it is reasonable to make the following assumption

AS2) Inter-state transitions only occur between adjacent states in the gain Markov model.

We define the level crossing rate (LCR) of a random process as the number of times it crosses a given level in the same direction within a given time interval. It is obtained in [12] that for isotropic scattering the LCR of $\eta(t) = g^2(t)$ at the level η_0 is given as

$$\beta_\eta(\eta_0) = \frac{\sqrt{2\pi} f_D \eta_0^{(L-1/2)}}{(L-1)! \exp(\eta_0)} \quad (11)$$

where f_D is the Doppler frequency. Following the method in [13], the probability of transition from state i to sate $i+1$ is related to the LCR of $g(t)$ as

$$P^{(g)}_{i,i+1} = P^{(g)}_{i+1,i} = \frac{T_s}{P^{(g)}_i}\beta_\eta(\hat{\eta}_i) \quad (12)$$

$$P^{(g)}_{i,i} = 1 - P^{(g)}_{i,i+1} - P^{(g)}_{i,i-1} \quad (13)$$

where T_s is the data symbol duration, $\beta_\eta(\hat{\eta}_i)$ is given in (11), and $P^{(g)}_i$ in (10).

3.2. Average Feedback Rates

We propose the feedback strategy that upon the occurrence of a transition a 1-bit feedback is sent to the transmitter for specifying one of the two adjacent states as the destination of the transition. This strategy guarantees the distortion of the CSI at the transmitter is caused only by quantization but not the time-variation of the channel. Using this strategy and given that the transition probability between two states is equal to the average number of transitions per data symbol period, we can obtain the average feedback rate as

$$R_g = \sum_{n=1}^{N_2} P_n^{(g)} (P_{n,n-1}^{(g)} + P_{n,n+1}^{(g)}) \quad \text{bits}/T_s \tag{14}$$

where $P_n^{(g)}$ is the steady-state probability given in (10) and $P_{n,n-1}^{(g)}$ and $P_{n,n+1}^{(g)}$ are transition probabilities given in (12).

4. FEEDBACK OF QUANTIZED BEAMFORMER

In this section, we consider the case of feeding back only the quantized channel beamformer, $\hat{\mathbf{u}}$. This case is applicable for a transmit beamforming system, where $\hat{\mathbf{u}}$ is used as the beamforming vector. In Section 4.1, we develop a Markov model for the quantized beamformer of a temporally correlated vector channel. This Markov model is used to analyze the limited-feedback rate in Section 4.2.

4.1. Beamformer Markov Model

In this section, we develop the *beamformer Markov model* that models the quantized beamformer of a temporally correlated vector channel. A codebook that defines the states of the beamformer Markov model is constructed in Section 4.1.1. Its properties are analyzed in Section 4.1.2.

4.1.1. State Space of the Beamformer Markov Model

We shall construct a codebook, the *beamformer codebook*, whose entries are the states of the beamformer Markov model. Let $\mathbb{U}$ denote the beamformer codebook and N_1 its cardinality. Hence, $\mathbb{U}$ contains N_1 unitary vectors: $\hat{\mathbf{u}}_1, \cdots, \hat{\mathbf{u}}_{N_1}$. The codebook construction involves quantization of the space of unitary vectors, in which the channel beamformer is defined. For quantization, we use the modified squared-error distortion function

$$d(\mathbf{u}; \hat{\mathbf{u}}) = \min_{\theta} \|\hat{\mathbf{u}} e^{j\theta} - \mathbf{u}\|^2. \tag{15}$$

where the phase θ is introduced to account for the phase-invariant property of a beamforming vector [10]. It can be shown that $e^{j\theta} = \hat{\mathbf{u}}^H \mathbf{u} / |\hat{\mathbf{u}}^H \mathbf{u}|$ and hence

$$d(\mathbf{u}; \hat{\mathbf{u}}) = 1 - |\hat{\mathbf{u}}^H \mathbf{u}|. \tag{16}$$

Since the above distortion function is identical to that used in [6] for constructing a beamforming codebook, we follow the same construction.

4.1.2. Steady-State and Transition Probabilities

For deriving the steady-state and transition probabilities, the following assumption based on the well-known Gersho's conjecture [17] are listed as follows.

AS3) The Voronoi regions for the code vectors in the codebook $\mathbb{U}$ have equal volumn.

The steady-state probability of each state of the beamformer Markov model is the probability that the channel beamformer is in this state. Using AS3), it is straightforward to show that the steady-state probabilities of the beamformer Markov states are all equal to $1/N_1$. First, the beamformer vector, $u(t)$, is isotropic, hence the steady-state probability of each Markov state is proportional to the area of the corresponding Voronoi cell. Second, from AS3), all Voronoi cells have equal volumn. From above observations, the steady-state probabilities of different states are hence equal to $1/N_1$. We summarize the above result in the following lemma.

Lemma 1. Given the assumption AS3), the stationary probabilities of the beamformer Markov model are equal, hence

$$P_i^{(s)} = \frac{1}{N} \quad \forall\, 1 \le i \le N. \tag{17}$$

As it is difficult to obtain closed-form expressions, we rely on the Monte Carlo method for computing the transition probabilities. A lemma that is useful for analyzing the average feedback rate in the next section is provided.

Lemma 2. Given the assumption AS3), for the beamformer Markov model, the probabilities of no transition are equal, hence

$$P_{11}^{(s)} = \cdots = P_{NN}^{(s)}. \tag{18}$$

4.2. Average Feedback Rate

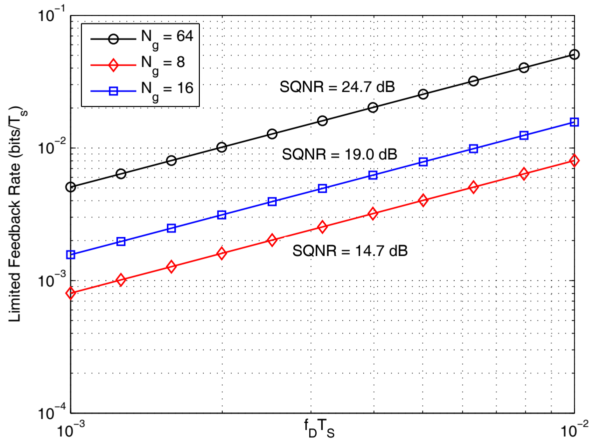
Similar feedback strategy as in Section 3.2 is adopted. The function of feedback is to inform the transmitter the sate change of the beamformer Markov model. To specify the new active sate, $\log_2 N_u$ bits are required for each feedback. The feedback frequency is related to inter-state transition probabilities in the same way as in Section 3.2. Hence, usig Lemma 1 and Lemma 2, the average limited feedback rate can be obtained as

$$R^{(s)}(\infty) = \log_2 N \left(1 - P_{11}^{(s)}\right) \quad \text{bits}/T_s. \tag{19}$$

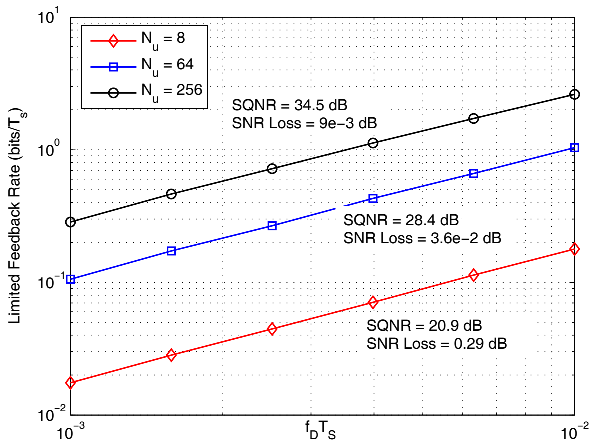
As mentioned, we rely on the Monte Carlo method for computing the probability $P_{11}^{(s)}$ in the above equation.

5. NUMERICAL RESULTS

We plot the feedback rate vs. normalized Doppler frequency ($f_D T_s$) curves in Fig 1(a) and Fig 1(b), corresponding to two cases: feedback of quantized channel gain and feedback of quantized channel direction. For Fig 1(a), the feedback rate is computed using (14) for a 2×2 MIMO system. Different sizes for the gain codebook are considered and the corresponding signal-to-quantization-noise ratios (SQNR) are shown. For Fig 1(b), the feedback rate for a 2×2 MIMO system is obtained using (19) where the $P_{11}^{(s)}$ is computed by Monte Carlo simulation. Different sizes of the beamformer codebook are considered and corresponding SQNR are shown. Also shown are the SNR loss due to quantized beamforming, which is defined as $\|\mathbf{H}\|_F^2 / \|\mathbf{H}\hat{\mathbf{u}}\|_F^2$. It can be observed that for both figures the feedback rates are linearly increasing the normalized Doppler frequency. Feedback rate curves similar to those in Fig 1 are useful for designing a MIMO limited-feedback system in three aspects. First, for a required SQNR, we can use the feedback-rate curves to determine the feedback rates for different Doppler frequencies. Second, for a given Doppler frequency, we can obtain from these curves SQNR's achieved by various feedback rates. Third, for a fixed feedback rate, we can use these curves to evaluate the impact of Doppler frequency on the SQNR.



(a)



(b)

Fig. 1. (a) Feedback Rate for Quantized Gain (2×2); (b) Feedback Rate for Quantized Beamformer (2×2).

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