LARGEST EIGENVALUE STATISTICS OF DOUBLE-CORRELATED COMPLEX WISHART MATRICES AND MIMO-MRC

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ABSTRACT

This paper considers multiple-input multiple-output (MIMO) antenna systems employing transmit beamforming (BF) with maximum ratio combining (MRC) receivers. Rayleigh fading environments are considered, with both transmit and receive spatial correlation. Exact expressions are presented for the probability density function (p.d.f.) of the output signal-to-noise ratio (SNR), as well as the system outage probability. The results are based on efficient closed-form expressions which we derive for the p.d.f. and c.d.f. of the maximum eigenvalue of double-correlated complex Wishart matrices. The results are validated through comparison with Monte-Carlo simulations, and used to examine the effect of spatial correlation on the SNR p.d.f. and the outage probability.

1. INTRODUCTION

Multiple-input multiple-output (MIMO) antenna technology can provide significant improvements in capacity [1, 2] and error performance over conventional single-antenna technology, without requiring extra power or bandwidth. Of the many practical MIMO transmission schemes that have been proposed, MIMO transmit beamforming (BF) with maximum-ratio combining (MRC) receivers [3] is particularly attractive when channel knowledge is available at both the transmitter and receiver. MIMO-BF systems provide robustness against the severe effects of fading by steering the transmitted signal along the maximum eigenmode of the MIMO channel, such that the signal-to-noise ratio (SNR) at the MRC output is maximized.

Recently, various authors have examined the performance of MIMO-BF in uncorrelated Rayleigh, semi-correlated Rayleigh, and uncorrelated Rician fading channels. In each case, the main challenge was to statistically characterize the SNR at the output of the MRC combiner. In [4, 5], uncorrelated Rayleigh fading was considered, and the output SNR statistical properties were derived based on maximum eigenvalue statistics of complex central Wishart matrices. In [6], these results were extended to semi-correlated Rayleigh channels, utilizing properties of semi-correlated Wishart matrices. In [7], the output SNR statistics in uncorrelated Rician channels were characterized by deriving maximum eigenvalue properties of complex noncentral Wishart matrices.

In practice, channels can exhibit double-sided correlation due to, for example, insufficient scattering around both the transmitter and receiver terminals, or to insufficiently spaced antennas (with respect to the wavelength of the signal). In these cases, there does not appear to be any analytic MIMO-BF performance results.

In this paper we statistically characterize the output SNR of MIMO-BF in practical double-sided correlated Rayleigh channels. We find that the SNR depends on the maximum eigenvalue statistics of double-correlated complex Wishart matrices. In [8] the joint probability density function (p.d.f.) of the eigenvalues of these matrices was derived in terms of hypergeometric functions of three matrix arguments, and the marginal p.d.f. of an *arbitrary* unordered eigenvalue was derived in [9]. In this paper we derive new exact closed-form expressions for the p.d.f. and cumulative distribution function (c.d.f.) of the maximum eigenvalue of double-correlated complex Wishart matrices. Based on these results, we then present expressions for the p.d.f. of the MIMO-BF output SNR, as well as the system outage probability. The expressions are verified through comparison with Monte-Carlo simulations, and used to examine the impact of correlation on both the SNR p.d.f. and outage probability.

2. MIMO BEAMFORMING SYSTEM MODEL

Consider a MIMO-BF system with N_t transmit and N_r receive antennas, where the $N_r \times 1$ received signal vector is

$$\mathbf{r} = \sqrt{\bar{\gamma}}\mathbf{H}\mathbf{w}x + \mathbf{n} \tag{1}$$

where x is the transmitted symbol with $E\left[|x|^2\right] = 1$, $\mathbf{w}$ is the beamforming vector (specified below) with $E\left[\|\mathbf{w}\|^2\right] = 1$, $\mathbf{n}$ is noise $\sim \mathcal{CN}_{N_r,1}\left(\mathbf{0}_{N_r\times 1}, \mathbf{I}_{N_r}\right)$, and $\bar{\gamma}$ is the signal to noise ratio (SNR). Also, $\mathbf{H}$ is the $N_r \times N_t$ channel matrix, assumed to be flat spatially-correlated Rayleigh fading, and is decomposed according to the common *kronecker* structure (as in [2, 8, 10], among others) as

$$\mathbf{H} = \mathbf{R}^{\frac{1}{2}}\mathbf{H}_w\mathbf{S}^{\frac{1}{2}} \sim \mathcal{CN}_{N_r,N_t}\left(\mathbf{0}_{N_r\times N_t}, \mathbf{R}\otimes\mathbf{S}\right) \tag{2}$$

where $\mathbf{R}$ and $\mathbf{S}$ are the receive and transmit correlation matrices respectively, satisfying $\operatorname{tr}(\mathbf{R}) = N_r$ and $\operatorname{tr}(\mathbf{S}) = N_t$, and $\mathbf{H}_w \sim \mathcal{CN}_{N_r,N_t}\left(\mathbf{0}_{N_r\times N_t}, \mathbf{I}_{N_r}\otimes\mathbf{I}_{N_t}\right)$.

The receiver employs the principle of MRC to give

$$\mathbf{z} = \mathbf{w}^\dagger\mathbf{H}^\dagger\mathbf{r} = \sqrt{\bar{\gamma}}\mathbf{w}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w}x + \mathbf{w}^\dagger\mathbf{H}^\dagger\mathbf{n} \tag{3}$$

Therefore, the SNR at the output of the combiner is easily derived as

$$\gamma = \bar{\gamma}\mathbf{w}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w} \tag{4}$$

The BF vector $\mathbf{w}$ is chosen to maximize this instantaneous SNR, thereby minimizing the error probability. Based on this criterion, it is well known that the optimum BF vector $\mathbf{w}_{\text{opt}}$ is the eigenvector corresponding to the maximum eigenvalue λ_m of $\mathbf{H}^\dagger\mathbf{H}$. In this case, the beamformed SNR (4) becomes

$$\gamma = \bar{\gamma}\mathbf{w}_{\text{opt}}^\dagger\mathbf{H}^\dagger\mathbf{H}\mathbf{w}_{\text{opt}} = \bar{\gamma}\lambda_m \tag{5}$$

Clearly the SNR (and therefore the performance) of MIMO-BF depends directly on the statistical properties of λ_m, which we consider in the following section. Note first that the following section considers full rank matrices, and as such, when $N_r \geq N_t$ we will use the results to analyze $\mathbf{H}^\dagger\mathbf{H}$, and when $N_r < N_t$ we will use the results to analyze $\mathbf{H}^\dagger\mathbf{H}$ (since λ_m is the maximum eigenvalue of both $\mathbf{H}^\dagger\mathbf{H}$ and $\mathbf{H}\mathbf{H}^\dagger$).

3. LARGEST EIGENVALUE STATISTICS OF DOUBLE-CORRELATED COMPLEX WISHART MATRICES

3.1. Cumulative Distribution Function (C.D.F.)

The following theorem presents the c.d.f. of the maximum eigenvalue of double-correlated complex Wishart matrices. This will be used for deriving the outage probability of MIMO-BF in double-correlated Rayleigh channels.

Theorem 1 *Let $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{0}_{m\times n}, \mathbf{\Sigma} \otimes \mathbf{\Omega})$, where $n \leq m$, and $\mathbf{\Omega} \in \mathcal{C}^{n\times n}$ and $\mathbf{\Sigma} \in \mathcal{C}^{m\times m}$ are Hermitian positive-definite matrices with eigenvalues $\omega_1 < \ldots < \omega_n$ and $\sigma_1 < \ldots < \sigma_m$ respectively. Then the c.d.f. of the maximum eigenvalue λ_m of the double-correlated complex Wishart matrix $\mathbf{X}^\dagger\mathbf{X}$ is given by*

$$F_{\lambda_m}(x) = \frac{(-1)^n \Gamma_n(n) \det(\mathbf{\Omega})^{n-1} \det(\mathbf{\Sigma})^{m-1} \det(\tilde{\mathbf{\Psi}}(x))}{\Delta_n(\mathbf{\Omega})\Delta_m(\mathbf{\Sigma})(-x)^{n(n-1)/2}} \quad (6)$$

where $\Gamma_n(\cdot)$ is the normalized complex multivariate gamma function, defined as[1]

$$\Gamma_n(n) = \prod_{i=1}^{n} \Gamma(n-i+1)\ . \quad (7)$$

and $\Delta_m(\cdot)$ is a Vandermonde determinant in the eigenvalues of the m-dimensional matrix argument, given by

$$\Delta_m(\mathbf{\Sigma}) = \prod_{i<j}^{m} (\sigma_j - \sigma_i) \quad (8)$$

Also, $\tilde{\mathbf{\Psi}}(x)$ is an $m \times m$ matrix with $(i,j)^{\text{th}}$ element

$$\left(\tilde{\mathbf{\Psi}}(x)\right)_{i,j} = \begin{cases} \left(\frac{1}{\sigma_j}\right)^{m-i} & \text{for i} \leq \tau \\ e^{-\frac{x}{\omega_{i-\tau}\sigma_j}} \mathcal{P}\left(m; -\frac{x}{\omega_{i-\tau}\sigma_j}\right) & \text{for i} > \tau \end{cases} \quad (9)$$

where $\tau = m - n$, and $\mathcal{P}(\ell; y) = 1 - e^{-y} \sum_{k=0}^{\ell-1} y^k/k!$ is the incomplete gamma function.

Proof: First consider the case of square random matrices $\mathbf{X} \sim \mathcal{CN}_{m,m}(\mathbf{0}_{m\times m}, \mathbf{\Sigma} \otimes \mathbf{\Omega})$, and let $\lambda_1 < \ldots < \lambda_m$ be the non-zero eigenvalues of $\mathbf{X}^\dagger\mathbf{X}$. The c.d.f. of λ_m is obtained using

$$F_{\lambda_m}(x) = \int_{\mathcal{D}} f(\mathbf{\Lambda})\, d\mathbf{\Lambda} \quad (10)$$

where $\mathbf{\Lambda} = \text{diag}\{\lambda_1, \ldots, \lambda_\text{m}\}$, $f(\mathbf{\Lambda})$ is the joint p.d.f. of $\lambda_1, \ldots, \lambda_m$, and $\mathcal{D} = \{0 \leq \lambda_1 \leq \ldots \leq \lambda_m < x\}$. It was shown in [8] that

$$f(\mathbf{\Lambda}) = \frac{{}_0F_0\left(-\mathbf{\Omega}^{-1}, \mathbf{\Sigma}^{-1}, \mathbf{\Lambda}\right)\Delta_m(\mathbf{\Lambda})^2}{\Gamma_m(m)^2 \det(\mathbf{\Omega})^m \det(\mathbf{\Sigma})^m} \quad (11)$$

[1]Note that this is related to the standard complex multivariate gamma function $\tilde{\Gamma}_n(n)$ (as defined in [11]) via $\Gamma_n(n) = \pi^{-n(n-1)/2}\tilde{\Gamma}_n(n)$.

where ${}_0F_0(\cdot;\cdot;\cdot)$ is a complex hypergeometric function of three matrix arguments. To evaluate the integral in (10) we first expand ${}_0F_0(\cdot)$ in complex zonal polynomials [11] as

$${}_0F_0\left(-\mathbf{\Omega}^{-1}, \mathbf{\Sigma}^{-1}, \mathbf{\Lambda}\right) = \sum_{k=0}^{\infty}\sum_{\mathcal{K}} \frac{\tilde{C}_\mathcal{K}(-\mathbf{\Omega}^{-1})\tilde{C}_\mathcal{K}(\mathbf{\Sigma}^{-1})\tilde{C}_\mathcal{K}(\mathbf{\Lambda})}{k!\tilde{C}_\mathcal{K}(\mathbf{I}_m)^2} \quad (12)$$

where the inner sum is over all partitions $\mathcal{K} = (k_1, \ldots, k_m)$ with $k_1 \geq k_2 \geq \ldots k_m \geq 0$, and $k_1 + \ldots + k_m = k$. Using the character representation for complex zonal polynomials [11], along with Weyl's formula [12], and changing from $\mathcal{K}$ to *strictly ordered* partitions $\mathcal{K}_o = (\tilde{k}_1, \ldots, \tilde{k}_m)$ with $\tilde{k}_1 > \ldots > \tilde{k}_m \geq 0$ (i.e. such that $k_i + m - i \to \tilde{k}_i$), we obtain[2]

$$\begin{aligned}{}_0F_0\left(-\mathbf{\Omega}^{-1}, \mathbf{\Sigma}^{-1}, \mathbf{\Lambda}\right) &= \frac{\Gamma_m(m)^2}{\prod_{i<j}^m\left(\frac{1}{\omega_j} - \frac{1}{\omega_i}\right)\prod_{i<j}^m\left(\frac{1}{\sigma_i} - \frac{1}{\sigma_j}\right)} \\ &\times \sum_{k=0}^{\infty}\sum_{\mathcal{K}_o} \frac{\det\left(\left(-\frac{1}{\omega_i}\right)^{\tilde{k}_j}\right)\det\left(\left(\frac{1}{\sigma_i}\right)^{\tilde{k}_j}\right)\det\left(\lambda_i^{\tilde{k}_j}\right)}{\left(\prod_{i=1}^m \tilde{k}_i!\right)\Delta_m(\mathcal{K}_o)\,\Delta_m(\mathbf{\Lambda})}\ . \end{aligned} \quad (13)$$

Substituting (13) into (11) and simplifying yields

$$\begin{aligned} f(\mathbf{\Lambda}) &= \frac{(-1)^{m(m-1)/2}}{\det(\mathbf{\Omega})\det(\mathbf{\Sigma})\Delta_m(\mathbf{\Omega})\Delta_m(\mathbf{\Sigma})} \\ &\times \sum_{k=0}^{\infty}\sum_{\mathcal{K}_o} \frac{\det\left(\left(-\frac{1}{\omega_i}\right)^{\tilde{k}_j}\right)\det\left(\left(\frac{1}{\sigma_i}\right)^{\tilde{k}_j}\right)\det\left(\lambda_i^{\tilde{k}_j}\right)\Delta_m(\mathbf{\Lambda})}{\left(\prod_{i=1}^m \tilde{k}_i!\right)\Delta_m(\mathcal{K}_o)}\ . \end{aligned} \quad (14)$$

Next, we substitute this into (10) to obtain

$$\begin{aligned} F_{\lambda_m}(x) &= \frac{(-1)^{m(m-1)/2}}{\det(\mathbf{\Omega})\det(\mathbf{\Sigma})\Delta_m(\mathbf{\Omega})\Delta_m(\mathbf{\Sigma})} \\ &\times \sum_{k=0}^{\infty}\sum_{\mathcal{K}_o} \frac{\det\left(\left(-\frac{1}{\omega_i}\right)^{\tilde{k}_j}\right)\det\left(\left(\frac{1}{\sigma_i}\right)^{\tilde{k}_j}\right)\mathcal{I}}{\left(\prod_{i=1}^m \tilde{k}_i!\right)\Delta_m(\mathcal{K}_o)} \end{aligned} \quad (15)$$

where

$$\mathcal{I} = \int_{\mathcal{D}} \det\left(\lambda_i^{\tilde{k}_j}\right)\ \Delta_m(\mathbf{\Lambda})\, d\mathbf{\Lambda}\ . \quad (16)$$

Omitting details, this integral can be evaluated as

$$\mathcal{I} = x^{m(m+1)/2+k}\Delta_m(\mathcal{K}_o)\prod_{i,j=1}^{m}\left(\frac{1}{\tilde{k}_i + j}\right)\Gamma_m(m) \quad (17)$$

and, as such, (15) can be written as

$$\begin{aligned} F_{\lambda_m}(x) &= \frac{(-1)^{m(m-1)/2}\,\Gamma_m(m)x^{m(m+1)/2}}{\det(\mathbf{\Omega})\det(\mathbf{\Sigma})\,\Delta_m(\mathbf{\Omega})\,\Delta_m(\mathbf{\Sigma})} \\ &\times \sum_{k=0}^{\infty}\sum_{\mathcal{K}_o}\det\left(\left(-\frac{1}{\omega_i}\right)^{\tilde{k}_j}\right)\det\left(\left(\frac{1}{\sigma_i}\right)^{\tilde{k}_j}\right)\prod_{i=1}^{m} g(\tilde{k}_i) \end{aligned} \quad (18)$$

[2]Here we introduce the compact notation for the determinant of a matrix, written in terms of the $(i,j)^{\text{th}}$ element. Also, for convenience we use $\Delta_m(\mathcal{K}_o)$ to denote $\Delta_m(\text{diag}\{\mathcal{K}_\text{o}\})$.

where

$$g(\tilde{k}_i) = \frac{x^{\tilde{k}_i}}{\tilde{k}_i!} \prod_{j=1}^{m} \left(\frac{1}{\tilde{k}_i + j} \right) . \quad (19)$$

We now apply the Cauchy-Binet formula [13] to give

$$F_{\lambda_m}(x) = \frac{(-1)^{m(m-1)/2}\, \Gamma_m(m)\, x^{m(m+1)/2}}{\det(\mathbf{\Omega}) \det(\mathbf{\Sigma})\, \Delta_m(\mathbf{\Omega})\, \Delta_m(\mathbf{\Sigma})} \times \det \left(\sum_{k=0}^{\infty} \left(-\frac{1}{\omega_i \sigma_j} \right)^k g(k) \right) \quad (20)$$

and perform some manipulations to remove the infinite sum, to give

$$F_{\lambda_m}(x) = \frac{(-1)^m\, \Gamma_m(m) \det(\mathbf{\Omega})^{m-1} \det(\mathbf{\Sigma})^{m-1}}{\Delta_m(\mathbf{\Omega})\, \Delta_m(\mathbf{\Sigma})\, (-x)^{m(m-1)/2}} \times \det \left(e^{-\frac{x}{\omega_i \sigma_j}} \mathcal{P} \left(m; -\frac{x}{\omega_i \sigma_j} \right) \right) . \quad (21)$$

This establishes the result for square matrices. To obtain the result for rectangular matrices (i.e. for $n < m$), we follow the approach of [9], and take limits of (21) as the eigenvalues $\omega_1 \to 0, \ldots, \omega_\tau \to 0$, to obtain the desired result in (6). □

Note that a complete derivation will be presented in an extended journal version of this paper.

Corollary 1 *For the case $n = 2, m = 2$, (6) reduces to*

$$F_{\lambda_m}(x) = \frac{\omega_1 \omega_2 \sigma_1 \sigma_2}{x(\sigma_2 - \sigma_1)(\omega_2 - \omega_1)} \sum_{i=1}^{2} (-1)^i \times \prod_{j=1}^{2} \left(e^{-\frac{x}{\omega_{|i-j|+1}\sigma_j}} + \frac{x}{\omega_{|i-j|+1}\sigma_j} - 1 \right) . \quad (22)$$

3.2. Probability Density Function (P.D.F.)

The following theorem presents the p.d.f. of the maximum eigenvalue of double-correlated complex Wishart matrices. This will be used for deriving the p.d.f. of the output SNR of MIMO-BF in double-correlated Rayleigh channels.

Theorem 2 *Let $\mathbf{X} \sim \mathcal{CN}_{m,n}(\mathbf{0}_{m \times n}, \mathbf{\Sigma} \otimes \mathbf{\Omega})$, where $n \leq m$, and $\mathbf{\Omega} \in \mathcal{C}^{n \times n}$ and $\mathbf{\Sigma} \in \mathcal{C}^{m \times m}$ are Hermitian positive-definite matrices with eigenvalues $\omega_1 < \ldots < \omega_n$ and $\sigma_1 < \ldots < \sigma_m$ respectively. Then the p.d.f. of the maximum eigenvalue λ_m of the double-correlated complex Wishart matrix $\mathbf{X}^\dagger \mathbf{X}$ is given by*

$$f_{\lambda_m}(\lambda_m) = \frac{(-1)^{n+1} \Gamma_n(n) \, \det(\mathbf{\Omega})^{n-1} \det(\mathbf{\Sigma})^{m-1}}{\Delta_n(\mathbf{\Omega})\, \Delta_m(\mathbf{\Sigma})\, (-\lambda_m)^{n(n-1)/2}} \times \left(\frac{n(n-1) \det(\tilde{\mathbf{\Psi}}(\lambda_m))}{2\lambda_m} + \sum_{\ell=\tau+1}^{m} \det(\tilde{\mathbf{\Psi}}_\ell(\lambda_m)) \right) \quad (23)$$

where $\tilde{\mathbf{\Psi}}_\ell(\lambda_m)$ is an $m \times m$ matrix with $(i,j)^{\text{th}}$ element

$$\left(\tilde{\mathbf{\Psi}}_\ell(\lambda_m) \right)_{i,j} = \begin{cases} \left(\tilde{\mathbf{\Psi}}(\lambda_m) \right)_{i,j} & \text{for i} \neq \ell \\ \frac{e^{-\frac{\lambda_m}{\omega_{i-\tau}\sigma_j}}}{\omega_{i-\tau}\sigma_j} \mathcal{P}\left(m-1; \frac{-\lambda_m}{\omega_{i-\tau}\sigma_j} \right) & \text{for i} = \ell \end{cases} \quad (24)$$

and where $\left(\tilde{\mathbf{\Psi}}(\lambda_m) \right)_{i,j}$ is defined in (9).

Proof: The result follows by differentiating (6) with respect to x, using a well-known formula for the derivative of a determinant. □

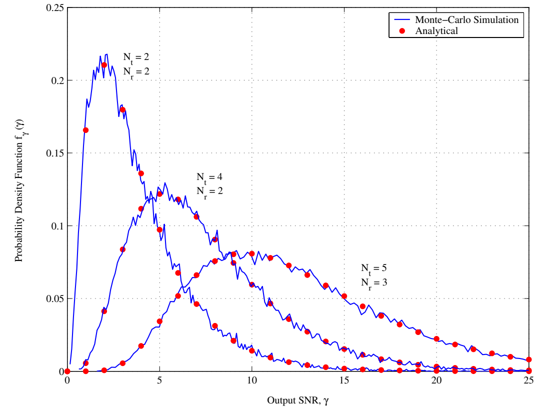

Fig. 1. P.d.f. of the output SNR of MIMO-BF in double-correlated Rayleigh channels, for $\bar{\gamma} = 0$ dB. Correlation parameters are $\theta_r = \theta_t = \frac{\pi}{2}$, $d = \frac{1}{2}$, $\sigma_r^2 = \frac{\pi}{64}$, and $\sigma_t^2 = \frac{\pi}{16}$.

4. STATISTICAL CHARACTERIZATION OF MIMO-BF OUTPUT SNR IN DOUBLE-CORRELATED CHANNELS

We now characterize the statistics of the SNR of MIMO-BF. Consider the case $N_r \geq N_t$ where Theorems 1 and 2 apply, with $\mathbf{X}$, n, m, $\mathbf{\Omega}$, and $\mathbf{\Sigma}$ corresponding to $\mathbf{H}$, N_t, N_r, $\mathbf{S}$, and $\mathbf{R}$ respectively[3]. Using (5), and making a simple change of variables to (23) we obtain the p.d.f. of γ given by

$$f_\gamma(\gamma) = \frac{\Gamma_{N_t}(N_t) \, \det(\mathbf{S})^{N_t-1} \det(\mathbf{R})^{N_r-1}}{(-1)^{N_t} \Delta_{N_t}(\mathbf{S})\, \Delta_{N_r}(\mathbf{R})} \left(-\frac{\bar{\gamma}}{\gamma} \right)^{N_t(N_t-1)/2} \times \left(\frac{N_t(1-N_t) \det \left(\tilde{\mathbf{\Psi}} \left(\frac{\gamma}{\bar{\gamma}} \right) \right) \bar{\gamma}}{2\gamma} + \sum_{\ell=\tau+1}^{N_r} \det \left(\tilde{\mathbf{\Psi}}_\ell \left(\frac{\gamma}{\bar{\gamma}} \right) \right) \right) . \quad (25)$$

The outage probability of MIMO-BF is defined as the probability that the SNR γ drops below a certain threshold γ_{th}, and hence is obtained directly from the c.d.f. of γ. Using (6), we evaluate this c.d.f. as

$$F_\gamma(\gamma_{\text{th}}) = \Pr(\gamma \leq \gamma_{\text{th}}) = F_{\lambda_m} \left(\frac{\gamma_{\text{th}}}{\bar{\gamma}} \right) = \frac{\Gamma_{N_t}(N_t) \, \det(\mathbf{S})^{N_t-1} \det(\mathbf{R})^{N_r-1} \det \left(\tilde{\mathbf{\Psi}} \left(\frac{\gamma_{\text{th}}}{\bar{\gamma}} \right) \right)}{(-1)^{N_t}\, \Delta_{N_t}(\mathbf{S})\, \Delta_{N_r}(\mathbf{R}) \left(-\frac{\gamma_{\text{th}}}{\bar{\gamma}} \right)^{N_t(N_t-1)/2}} . \quad (26)$$

5. NUMERICAL RESULTS

Fig. 1 shows the p.d.f. of the output SNR of MIMO-BF with various antenna configurations in double-correlated Rayleigh channels. The channels are constructed based on the correlated model from [10]. The analytical curves are based on (25), and clearly agree with the Monte-Carlo simulated p.d.f.s. Moreover, we observe that both the mean and variance of the SNR increase with the number of antennas.

Fig. 2 shows the (analytical) p.d.f. of the output SNR, comparing various correlation scenarios. We see that for 4×4 antennas,

[3] Note that all the results in this section apply directly to the other case of $N_r < N_t$, simply by swapping N_t and N_r, and also $\mathbf{S}$ and $\mathbf{R}$.

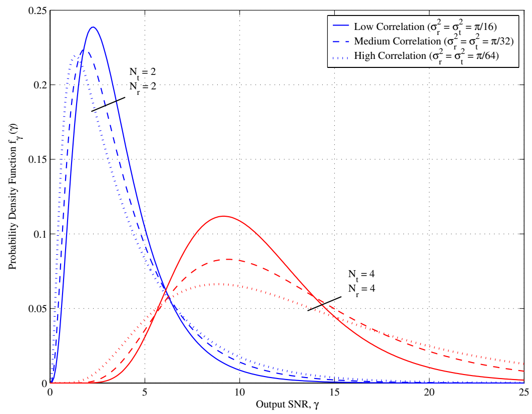

Fig. 2. P.d.f. of the output SNR of MIMO-BF in various double-correlated Rayleigh channels, for $\bar{\gamma} = 0$ dB.

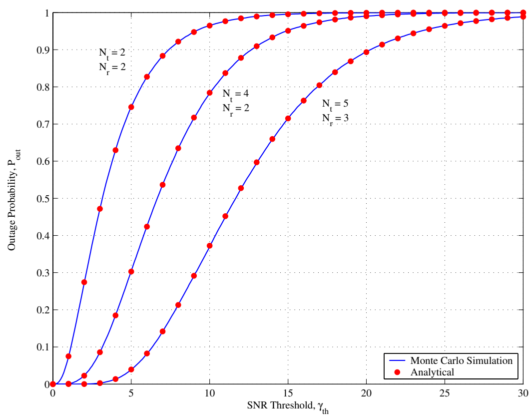

Fig. 3. Outage probability of MIMO-BF in double-correlated Rayleigh channels, for $\bar{\gamma} = 0$ dB. Correlation parameters are $\theta_r = \theta_t = \frac{\pi}{2}$, $d = \frac{1}{2}$, $\sigma_r^2 = \frac{\pi}{64}$, and $\sigma_t^2 = \frac{\pi}{16}$.

extra correlation increases the spread of the SNR around the mean. This agrees with previous observations for semi-correlated channels, given in [6]. For the 2×2 case, the correlation clearly has less effect.

Fig. 3 shows the outage probability of MIMO-BF, with the same antenna and correlation parameters as for Fig. 1. The analytical curves are based on (26), and agree precisely with Monte-Carlo simulated curves. We see that the outage probability is significantly improved as the number of antennas are increased.

Fig. 4 shows (analytical) outage probability curves, comparing different correlation scenarios. We see that for both antenna configurations, the correlation increases the outage probability at low SNR thresholds, and decreases the outage probability (thereby improving system performance) at high SNR thresholds. This agrees with previous semi-correlated results from [6]. Moreover, we see that the cross-over point of the different correlated curves occurs at lower outage probabilities as the numbers of antennas increase.

6. CONCLUSION

We have presented exact closed-form expressions for the p.d.f. of the output SNR as well as the outage probability of MIMO-BF with

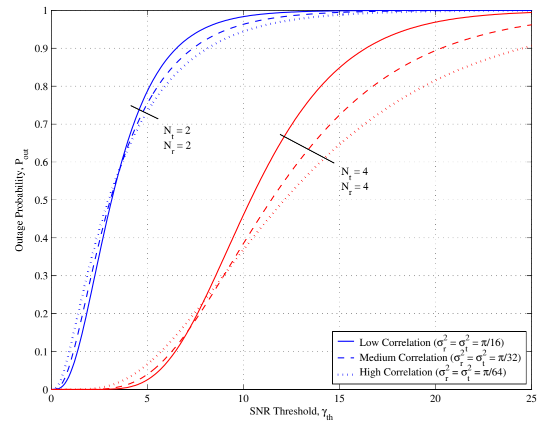

Fig. 4. Outage probability of MIMO-BF in various double-correlated Rayleigh channels, for $\bar{\gamma} = 0$ dB.

MRC in double-correlated Rayleigh channels. Based on the analytical results, we examined the effect of spatial correlation on the SNR.

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