# A STOCHASTIC SEARCH APPROACH FOR UAV TRAJECTORY PLANNING IN LOCALIZATION PROBLEMS

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## ABSTRACT

We discuss the off-line and on-line aspects of trajectory planning in bearings-only localization. Assuming that there are $m(\geq 1)$ moveable sensors (e.g. UAVs), which fly in closed trajectories, the aim is to determine the optimal shape of the trajectory. We investigate the properties of closed optimal trajectories in the off-line problem and show that these solutions are invariant under a scaling transformation of the problem parameters. This result is used to numerically derive a set of solutions for the normalized parameters. These solutions are then used in a stochastic search algorithm which randomly explores the trajectories but spends the largest amount of time in the optimal trajectory.

## 1. INTRODUCTION

Consider a stationary target at an unknown location in a known and bounded area. There are $m(\geq 1)$ Unmanned Aerial Vehicles (UAVs) available to locate this target. The UAVs can move within the boundaries of this area under certain constraints and collect noisy measurements from some parameter that is related to the position of the target. We aim to compute the optimal trajectory for each UAV from time 1 to $N$, $\Theta_i = \{X_s^i(k)\}_{k=1}^N$, where $1 \leq i \leq m$ such that at the end of this period, a cost function reflecting the accuracy of our estimation of the position of this target is optimized. Formally stated, we are interested in

$$\begin{aligned} \underset{\Theta_1,\cdots,\Theta_m}{\arg\min} \quad & g(E\{[P - h(Z^{1:N}(P, \Theta_1, \ldots, \Theta_m))]^T \\ & [P - h(Z^{1:N}(P, \Theta_1, \ldots, \Theta_m))]\}) \qquad (1) \\ \text{subject to} \quad & \Theta_i \in \mathbb{U}_i \quad \text{for} \quad 1 \leq i \leq m, \end{aligned}$$

where

- $P$ is the position vector of the target. Following the non-Bayesian approach [1], we assume $P$ is a fixed and non-random variable;
- $\mathbb{U}_i$ defines the set of constraints imposed on the trajectory of the $i$-th sensor;
- $Z^{1:N}$ is the set of noisy measurements from all $m$ UAVs from time 1 to time $N$. Measurements are functions of $P$ and $\Theta_1, \ldots, \Theta_m$. Expectation in (1) is taken with respect to the probability distribution of $Z^{1:N}$;
- Function $h$ is an efficient and unbiased estimator of $P$. It is assumed that the mechanism by which measurements are fused is given to us. Function $g$ is a real-valued function of its agrement. Since the covariance matrix can be multi-dimensional, function $g$ is mainly used to establish an ordering relation between the performance of various trajectories. $g$ can be the determinant, trace, the largest eigenvalue or an element of the covariance matrix.

We note that in (1), $P$ is not known so the full structure of the cost function is not known to us and a closed-form solution cannot be found. However, a relevant question to ask is even if $P$ is known, which trajectories are the solutions of (1). The answer to this question leads to minimizing the Cramer-Lower Bound (CRLB) of estimation error covariance. It should be pointed out that even finding an analytical solution for this off-line optimization problem is not feasible in most cases and one has to only rely on numerical methods.

The results of the off-line optimization can be used in a stochastic search method to solve (1) in real time if it is ensured that the underlying process is stationary. In particular, we devise a modified version of the search algorithm outlined in [2] for this purpose. The proposed algorithm is such that while all candidate trajectories are explored randomly, the largest amount of time is spent on travelling the global optimal trajectory.

Bearings-only localization is one case in which sensor trajectory planning is important. In bearings-only localization, the target emits some kind of energy (sound, electromagnetic, etc.) to its surrounding environment and sensors measure the angle of arrival of the signal and estimate the relative range of the target. The problem of sensor trajectory planning for bearings-only localization has been mainly studied in the literature as an off-line optimization problem [3, 4, 5]. We extend these results to the on-line optimization problem. This requires that the initial relative range of the target from the sensor remains fixed at each iteration of the stochastic search algorithm and therefore only closed trajectories are allowed. A prime example of the application of this method is for UAVs, which are equipped with monocular-vision system and sent on a mission to monitor a target and estimate its position.

This paper has four sections. In Section 2, the optimization of the CRLB for closed trajectories is discussed, where we show the optimal trajectory is invariant to a special scaling transformation of the problem parameters and derive the numerical result for the normalized parameters. In Section 3, the stochastic search algorithm is discussed. Finally, in Section 4, some concluding remarks are made.

## 2. OPTIMIZING THE CRLB FOR BEARINGS-ONLY LOCALIZATION

We construct a coordinate system for the bearings-only localization problem following the arrangement shown in Fig.1. The initial location of the sensor is defined as the origin. The $x$-axis is defined as the initial direction of arrival of the signal from the sensor and the $y$-axis is defined such that a right-handed coordinate system is built. It is assumed that the target is located at distance $x_t$ on the $x$-axis where $r_{min} < x_t < r_{max}$. During the observation period, the sensor travels with its maximum speed $V$ and its trajectory is defined by a sequence of course inputs (see below for definition). The sensor must return to the origin at end of the observation period and cannot reach the target during the observation period, therefore $V\Delta TN < 2r_{min}$, where $\Delta T$ is the sampling period. With this formulation, problem (1) can be stated as follows for the bearings-only localization problem using a single sensor. Find

$$\arg\min_{U_s} \quad E\{[x_t - \hat{x}_t(Z^{1:N}(x_t, \Theta(U_s)))]^2\}, \qquad (2)$$

where

- $\Theta(U_s) = \{X_s(k)\}_{k=1}^N = \{[x_s(k), y_s(k)]^T\}_{k=1}^N$ is the sensor trajectory with boundary conditions $x_s(0) = y_s(0) = x_s(N) = y_s(N) = 0$. The trajectory evolves according to

$$\begin{bmatrix} x_s(k+1) \\ y_s(k+1) \end{bmatrix} = \begin{bmatrix} x_s(k) \\ y_s(k) \end{bmatrix} + V\Delta T \begin{bmatrix} \cos u^s(k) \\ \sin u_s(k) \end{bmatrix}, \quad (3)$$

where $U_s = \{u_s(k)\}_{k=1}^N$ is the sequence of course commands;

- $\hat{x}_o$ is the estimator of $x_t$;
- $Z^{1:N}(x_t, \Theta(U_s)) = \{z(k)\}_{k=1}^N$ is the set of noisy bearing measurements where bearing, $\beta(k)$, is defined as the angle that the line connecting the sensor to the target makes with the $x$-axis

$$\beta(k) = \text{atan2}(0 - y_s(k), x_t - x_s(k)), \qquad (4)$$

and $z(k)$ is defined as

$$z(k) = \beta(k) + w(k), \qquad (5)$$

where atan2$(y, x)$ is the arc tangent of $y/x$ in the interval $[-\pi, \pi]$ and the sign of atan2 is determined by the sign of $y$ and $\{w(k)\}_{k=1}^N$ is a sequence of zero mean independent Gaussian noise with variance $\sigma_w^2$.

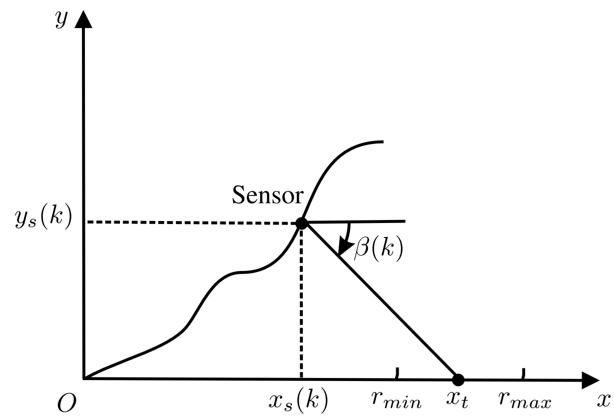


**Fig. 1**. The geometry of bearings-only localization.

In finding the optimal sensor trajectory for a given $x_t$, we assume that a Maximum Likelihood Estimator (MLE) is used. From (5), we note that the probability density function of $Z^{1:N}$ for a given $x_t$ is

$$p(Z^{1:N}; x_t) = \prod_{k=1}^{N} \frac{1}{\sigma_w\sqrt{2\pi}} \exp(-\frac{(z(k) - \beta(k))^2}{2\sigma_w^2}). \quad (6)$$

The Cramer-Rao Lower Bound (CRLB) is defined as the inverse of the Fisher Information Matrix (FIM) where the FIM for a given sequence of course commands is defined as

$$J(U_s) = E\{[\frac{\partial \ln p(Z^{1:N}; x_t)}{\partial x_t}]^2\}. \qquad (7)$$

With some extra work to combine (6) and (7), which is outlined in [6], a direct expression for $J(U_s)$ can be obtained (see below). As a result, problem (2) is converted to the following deterministic optimization problem. Find

$$\arg\min_{U_s} J(U_s) = \frac{-1}{\sigma_w^2} \sum_{k=1}^{N} \frac{y_s(k)^2}{[(x_t - x_s(k))^2 + y_s(k)^2]^2}, \qquad (8)$$

where $\{[x_s(k), y_s(k)]^T\}_{k=1}^N$ satisfy the constraints stated in (3). A continuous-time equivalent of (8) can be stated as finding

$$\begin{aligned} &\arg\min_{U_s} \quad J(U_s) = \frac{-1}{\sigma_w^2} \int_0^{T_f} \frac{y_s(t)^2 dt}{[(x_t - x_s(t))^2 + y_s(t)^2]^2} \\ &\text{subject to} \quad \dot{x_s}(t) = V\cos(U_s(t)), \quad \dot{y_s}(t) = V\sin(U_s(t)) \\ &\qquad x_s(0) = y_s(0) = x_s(T_f) = y_s(T_f) = 0, \end{aligned} \qquad (9)$$

where $T_f$ is the final time. Note that 0 and $T_f$ in the continuous case correspond to 1 and $N\Delta T$ in the discrete case, respectively. Problem (9) has the form of classical problems in optimal control theory and no analytical solution has been found for it. A solution is given in [3] for a rather simpler problem with no constraint on the final position of the sensor. Even for that case, a nonlinear root finding problem must be solved to obtain an analytical solution. However an important observation is made in [3] that the solution of (9) is unique for a given parameter $C$ known as the range-to-baseline ratio. $C$ is defined as

$$C = VT_f/x_t. \tag{10}$$

This is an important observation, since if one wants to rely on numerical methods to solve (9), instead of working with three different variables, mainly, $V$, $T_f$ and $x_t$, one needs to only work with $C$. This important fact is stated informally in [3]. We give a complete proof of it in the following proposition.

**Proposition.** *If $U_s(\xi)$ satisfies the necessary conditions of optimality in problem (9) for given values of $x_t, V$ and $T_f$, where $\xi = t/T_f$ and $0 \leq \xi \leq 1$, then $U_s(\xi')$ also satisfies the same optimality conditions for $x'_t, V'$ and $T'_f$, where $\xi' = t/T'_f$, as long as $C = VT_f/x_t = V'T'_f/x'_t$ holds.*

*Proof.* Problem (9) when reformulated as a function of $\xi$ has the following form

$$\begin{aligned} \arg\min_{U_s} \quad & J(U_s) = \frac{-T_f}{\sigma_w^2}\int_0^1 \frac{y_s(\alpha)^2 d\alpha}{[(x_t - x_s(\alpha))^2 + y_s(\alpha)^2]^2} \\ \text{subject to} \quad & \dot{x_s}(\xi) = VT_f\cos(U_s(\xi)), \\ & \dot{y_s}(\xi) = VT_f\sin(U_s(\xi)) \\ & x_s(0) = y_s(0) = x_s(1) = y_s(1) = 0. \end{aligned} \tag{11}$$

Let $x'_s(.)$ and $y'_s(.)$ be the optimal trajectory when the second set of parameters, $x'_t, V'$ and $T'_f$, are used. We note that

$$\begin{aligned} x'_s(\xi) &= \int_0^\xi V'T'_f\cos(U_s(\alpha))d\alpha = \frac{V'T'_f}{VT_f}x_s(\xi) \\ y'_s(\xi) &= \int_0^\xi V'T'_f\sin(U_s(\alpha))d\alpha = \frac{V'T'_f}{VT_f}y_s(\xi). \end{aligned} \tag{12}$$

We can immediately see from (12) that if the boundary conditions are satisfied for $x_s(.)$ and $y_s(.)$, they are also satisfied for $x'_s(.)$ and $y'_s(.)$. Next we note that the Hamiltonian for the original set of parameters is

$$\begin{aligned} \mathcal{H} = &-\frac{T_f}{\sigma_w^2}\frac{y_s(\xi)^2}{[(x_t - x_s(\xi))^2 + y_s(\xi)^2]^2} + \\ &\lambda_1(\xi)VT_f\cos(U_s(\xi)) + \lambda_2(\xi)VT_f\sin(U_s(\xi)) \end{aligned} \tag{13}$$

where $\lambda_1$ and $\lambda_2$ are the adjoint states satisfying

$$\begin{aligned} \dot{\lambda_1}(\xi) &= -\frac{\partial \mathcal{H}}{\partial x_s} = \frac{T_f}{\sigma_w^2}\frac{4y_s(\xi)^2(x_t - x_s(\xi))}{[(x_t - x_s(\xi))^2 + y_s(\xi)^2]^3} \\ \dot{\lambda_2}(\xi) &= -\frac{\partial \mathcal{H}}{\partial y_s} = \frac{T_f}{\sigma_w^2}\frac{2y_s(\xi)(x_t - x_s(\xi))^2 - y_s(\xi)^3}{[(x_t - x_s(\xi))^2 + y_s(\xi)^2]^3} \\ 0 &= \frac{\partial \mathcal{H}}{\partial U_s} = VT_f(-\lambda_1\sin(U_s(\xi)) + \lambda_2\cos(U_s(\xi))). \end{aligned} \tag{14}$$

Now, let $\lambda'_1$ and $\lambda'_2$ be the adjoint states of the secondary problem. In this case for $\lambda'_1$, we have

$$\begin{aligned} \dot{\lambda'_1}(\xi) &= \frac{T'_f}{\sigma_w^2}\frac{4y'_s(\xi)^2(x'_t - x'_s(\xi))}{[(x'_t - x'_s(\xi))^2 + y'_s(\xi)^2]^3} \\ &= (\frac{V'T'_f}{VT_f})^{1/3}\frac{T'_f}{\sigma_w^2}\frac{4y_s(\xi)^2((\frac{VT_f}{V'T'_f})x'_t - x_s(\xi))}{[((\frac{VT_f}{V'T'_f})x'_t - x_s(\xi))^2 + y_s(\xi)^2]^3} \\ &= (\frac{V'T'_f}{VT_f})^{1/3}\frac{T'_f}{\sigma_w^2}\frac{4y_s(\xi)^2((x_t - x_s(\xi))}{[(x_t - x_s(\xi))^2 + y_s(\xi)^2]^3} \\ &= (\frac{V'T'_f}{VT_f})^{1/3}\dot{\lambda_1} = B\dot{\lambda_1}, \end{aligned} \tag{15}$$

where in the second and third line, we have used (12) and the assumption of the proposition, respectively. Similarly, we can show $\dot{\lambda'_2}(\xi) = B\dot{\lambda_2}$. Now by letting $\lambda'_1(\xi) \triangleq B\lambda_1(\xi)$ and $\lambda'_2(\xi) \triangleq B\lambda_2(\xi)$, it follows from (14) that

$$0 = V'T'_f(-\lambda'_1\sin(U_s(\xi)) + \lambda'_2\cos(U_s(\xi)). \tag{16}$$

Hence, from (12), (15) and (16), we can see that $x'_s(.), y'_s(.)$, $\lambda'_1(.), \lambda'_2(.)$ and $U_s(.)$ satisfies all the boundary and optimality necessary conditions for the secondary problem. □

By the application of the above proposition, numerical methods such as the method of steepest descent [7] can be utilized to find the numerical solutions of (9) for various values of $C$. These results are shown in Fig. 2. Note that in this figure, the value of $C$ varies between 0 and 2 by changing the value of $x_t$. When $C = 2$, it means that if the sensor travels along the $x$-axis, it can reach the target and return to the origin by the end of the observation period. The results of Fig. 2 also suggest that the optimal trajectories can be approximated by a straight line. In other words, one can solve (11) for the following input

$$U_s(\xi) = \begin{cases} \phi & 0 \leq \xi \leq 1/2 \\ \phi - \pi & 1/2 < \xi \leq 1. \end{cases} \tag{17}$$

This will lead to a nonlinear root finding problem of which the answers are entered in Fig. 2 for the given values of $C$.

## 3. STOCHASTIC SEARCH ALGORITHM FOR BEARINGS-ONLY LOCALIZATION

Our stochastic search algorithm generates a Markov chain that spends most of its time at the optimal trajectory. The algorithm randomly selects one of the trajectories in Fig. 2. The sensor then travels this trajectory $l$ times where $l > 1$. At the end of each trip, the MLE estimates $x_t$. After $l$ times, the algorithm computes the sample variance of $\hat{x}_o$ for the travelled trajectory. Then another trajectory is selected and the sample variance for the new trajectory is computed. In the next step of the algorithm, whichever trajectory that had a lower sample variance in the previous step competes against another randomly selected trajectory among the family of trajectories including the winner trajectories. This way, at the cost of slower convergence, the search algorithm spends more time near the optimal solution and a trade-off can be made between exploitation and exploration. Overall, the This algorithm can be stated formally as follows.

**Step 0:** Let $\mathbb{P} = \{1, 2, \ldots, \mathcal{P}\}$ be the set of indices, each referring to one trajectory. Select a trajectory $p(0) \in \mathcal{P}$. Let $\mathcal{N}(p(0)) = 1$ and $\mathcal{N}(n) = 0$ for all $n \in \mathcal{P}$ and $n \neq p(0)$. Let $m = 0$ and $p^* = p(0)$, where $m$ represents the number of iterations of the algorithm and $p^*$ is the index to the optimal trajectory found by the algorithm. Go to step 1.

**Step 1:** Select another random trajectory $q(m)$ such that $q(m) = p(m)$ with probability $\frac{R}{Q}$ and $q(m) = n$, for all $n \in \mathcal{P}$ and $n \neq p(m)$, with probability $\frac{1}{Q}$, where $Q = P - 1 + R$ and $R$ is a chosen positive number. Go to step 2.

**Step 2:** Compute the sample variance of $\hat{x}_o$ for both $p(m)$ and $q(m)$. If $\sigma^2_{q(m)}(\hat{x}_o) > \sigma^2_{p(m)}(\hat{x}_o)$, then let $p(m+1) = q(m)$. Otherwise, let $p(m+1) = p(m)$. Go to step 3.

**Step 3:** Let $m = m + 1$ and $\mathcal{N}(p(m)) = \mathcal{N}(p(m)) + 1$. If $\mathcal{N}(p(m)) > \mathcal{N}(p^*)$, then let $p^* = p(m)$. Go to step 1.

We point out that the exploitation phase can occur in step 1 when $q(m) = p(m)$. Similar to the proof in [2], it can be shown the above algorithm converges almost surely to the optimal solution as long as the three assumptions stated there on the stochastic ordering of the candidate trajectories are stratified. We have not yet shown these assumptions hold for bearings-only localization but our simulation results show that the above algorithm usually converges to the optimal solution after 50 iterations when 8 to 10 trajectories are used with $\sigma_w^2 = (\pi/180°)^2$.

## 4. CONCLUSIONS

In this paper, we outlined a method for real time trajectory planning in bearings-only localization. By taking closed trajectories, it is ensured that the sensor does not "lose" the target by taking non-optimal trajectories in the absence of any information about the position of the target. Additionally if time is given, the sensor can eventually loop in the optimal trajectory and continuously monitor the target.

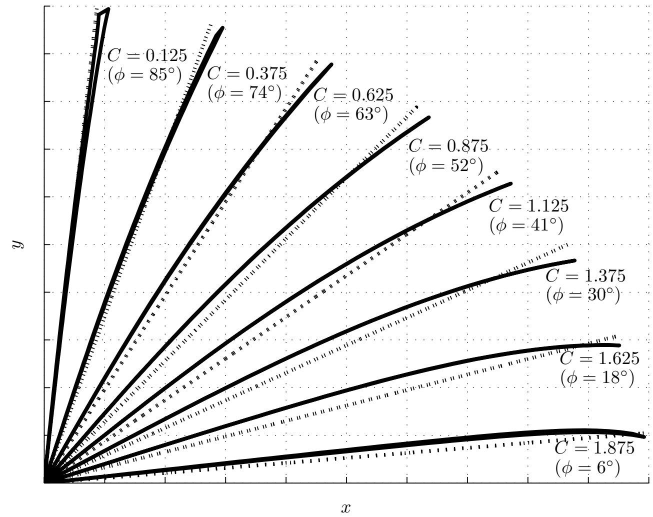


**Fig. 2**. Closed optimal sensor trajectories initiated and ended at the origin for various values of $C$.

## 5. REFERENCES


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