# New Designs of MAXFLAT FIR Halfband Low/High Pass Digital Filters with Narrow Transition Bands

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Abstract— MAXFLAT FIR digital filters have smoothest and the most accurate magnitude responses among all the available types of FIR digital filters. However, the transition bands of MAXFLAT filters are relatively wider, and this makes them unattractive in certain applications. Traditionally, low/high pass MAXFLAT FIR filters are designed to satisfy MAXFLAT constraints at ends of the frequency band. In this paper, we present two new designs of halfband low/high pass filters that satisfy these constraints at the middle of the pass and stop bands. The first design is obtained by solving a system of linear equations obtained by applying MAXFLAT constraints to the magnitude response of the filter, while the second design is a transformation of one of our previous designs of maximally linear digital differentiators. Design examples show that the transition bands of the presented designs are narrower as compared to traditional designs.

#### I. INTRODUCTION

Maximally flat (MAXFLT) finite impulse response (FIR) digital filters (DFs), introduced by Hermann [1] in 1971, are known for their design simplicity, accuracy and high stopband attenuations. The basic idea behind MAXFLAT designs is to force the magnitude response to be as close to the ideal as possible, at one or more fixed points in the frequency band. The resultant filters have highly smooth ripple-free magnitude responses in narrow regions centered at the points at which maximal flatness (MAXFLAT) constraints were applied. Classical MAXFLAT designs involve approximation of the desired frequency response by some suitable polynomial like a Hermite [1], Krawtchouk [2] or Bernstein polynomial [3] etc., which is then mapped to the filter function by certain transformations. The impulse response (IR) coefficients of the filters designed using these algorithms can be calculated using inverse discrete Fourier transform (IDFT).

Halfband low/high pass FIR DFs have cutoff frequency at the middle of the frequency band  $\omega = \pi /2$ , and are widely popular due to the fact that almost half of their IR coefficients are zeros, which leads to computationally efficient implementations. Halfband filters find several applications in filter banks, wavelets based compression and multirate techniques, and several designs and implementation tricks have been proposed for these filters including both MAXFLAT and ripple designs [4-11]. Gumacos [4] presented simplified expressions for the IR coefficients of MAXFLAT halfband designs in 1978. Samadi *et. al.* presented designs for generalized halfband MAXFLAT filters [5] and an efficient implementation of these filters was given by Pei *et. al.* in [6].

The transition bands of the existing MAXFLAT DFs are relatively wider and can be narrowed only by increasing the length of the filter. Almost all available MAXFLAT low / high pass designs satisfy the MAXFLAT constraints at zero frequency  $\omega = 0$  and the Nyquist frequency  $\omega = \pi$ . In this paper we present a new design for MAXFLAT halfband low/high pass filters which satisfy the MAXFLAT constraints at middle of their pass and stop bands, i.e., at  $\omega =$  $\pi/4$  and  $\omega = 3 \pi/4$ , and have relatively narrower transition bands as compared to existing MAXFLAT designs. These filters however have poor performance at  $\omega = 0$  and  $\omega = \pi$ .

We present another MAXFLAT halfband design, by transforming an existing design of digital differentiators (DDs) [12], having maximal linearity (MAXLIN) at  $\omega = \pi$ . Like above-mentioned new designs, these halfband designs also satisfy MAXFLAT constraints at the middle of their pass and stop bands; however their order of flatness is one less than the order of flatness of the above-mentioned designs based on MAXFLAT constraints. This extra degree of freedom makes the magnitude response smoother in the lower and upper ends of the frequency band, with almost no effect on the width of the transition band. Design examples are presented to compare the two proposed designs with each other and existing MAXFLAT halfband designs.

# II. MAXFLAT HALFBAND LOWPASS DIGITAL FILTERS

### A. Based on Direct Application of MAXFLAT Constraints

An ideal halfband lowpass magnitude response can be written as

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$$H(\omega) = \begin{cases} 1 & 0 \le \omega \le \pi / 2 \\ 0 & \pi / 2 \le \omega \le \pi. \end{cases}$$
(1)

Since the ideal responses cannot be achieved with filters of finite lengths, the MAXFLAT design criteria is to force the magnitude response, and as many as possible its higher derivatives, to have the values equal to the ideal at a single specified frequency  $\omega_0$ . In other words, for a filter to be MAXFLAT at a certain frequency  $\omega_0$ , lying in the passband, its magnitude response  $H(\omega)$  should satisfy the following conditions

$$H(\omega_0) = 1, \tag{2}$$

and

$$\frac{d^{i}H(\omega)}{d\omega^{i}}\bigg|_{\omega=\omega_{0}} = 0, 1 \le i \le L,$$
(3)

where L is a suitable number depending on the length of the filter.

It can be shown that the central coefficient of a halfband FIR DF is equal to 0.5 and all even indexed coefficients are zeros. Therefore magnitude response of a type I (odd number of even-symmetric coefficients) FIR DF of length 4N-1 can be expressed as

$$H(\omega) = 0.5 + 2\sum_{n=1}^{N} h_{2n-1} \cos(2n-1)\omega, \qquad (4)$$

where  $h_{n,} - 2N < n < 2N$  's are the IR coefficients and have even symmetry.

The existing MAXFLAT designs satisfy these constraints at the ends of the frequency band,  $\omega = 0$  and  $\omega = \pi$ . Here, we derive a new design satisfying the constraints at  $\omega = \pi / 4$  and  $\omega = 3 \pi / 4$ . The stopband of MAXFLAT halfband DFs is exactly an inverted mirror image of the passband, and therefore applying MAXFLAT constraints at a certain frequency  $\omega_0$  in the passband automatically satisfies the constraints at  $\pi - \omega_0$  in the stopband.

Applying the MAXFLAT constraints of Eqs. (2-3) to Eq. (4), at middle of the passband  $\omega = \pi / 4$ , we get the following set of linear equations:

$$\sum_{n=1}^{N} h_{2n-1} \cos((2n-1)\pi / 4) = 0.25$$

$$\sum_{n=1}^{N} (2n-1)^{i} (-1)^{n-1} h_{2n-1} \cos((2n-1)\pi / 4) = 0,$$

$$1 \le i \le N, i = odd$$
(5)

$$\sum_{n=1}^{N} (2n-1)^{i} h_{2n-1} \cos((2n-1)\pi / 4) = 0,$$
  
$$1 \le i \le N, i = even$$

The  $(N \ge N)$  coefficients matrix of these equations is Vandermonde matrix and therefore can be written in closed from, and this facilitates finding the solution of the equations. After some mathematical derivations, the complete set of IR coefficients **h** can be written in closed form as:

$$h_{0} = 1/2$$

$$h_{0} = 1/2 = \begin{cases} \frac{(-1)^{n-1}(2N-1)!!}{2^{N}\sqrt{2}(2n-1)(N-n)!!(N+n-2)!!} \\ N-n = even \\ \frac{(-1)^{n-1}(2N-1)!!}{2^{N}\sqrt{2}(2n-1)(N-n-1)!!(N+n-1)!!} \\ N-n = odd \\ 1 \le n \le N \end{cases}$$

$$h_{+2n} = 0, \ 1 \le n < N \end{cases}$$
(6)

where the double factorial of an integer m is defined as

$$m!! = m(m-2)(m-4)...(\geq 1)$$
.

Magnitude response of a halfband lowpass filter designed this way is shown in Fig. 1, along with a traditional filter satisfying MAXFLAT constraints at  $\omega = 0$ . It can be seen that the transition band of the proposed filter is relatively narrow but its performance has deteriorated at the ends of the frequency band. Due to this reason, it can be used only in very limited applications, where useful signal contents are not very close to the ends of the frequency bands. In the next subsection, we derive another MAXFLAT halfband design with narrow transition bands and better performance at the ends of the frequency band.

### B. Based on Transformation of MAXLIN DDs

Magnitude response of an ideal DD for  $0 \le \omega \le 2\pi$  can be written as

$$D(\omega) = \begin{cases} \omega & 0 \le \omega \le \pi \\ \pi - \omega & \pi \le \omega \le 2\pi. \end{cases}$$
(7)

For a filter to be MAXLIN at a certain frequency  $0 \le \omega_0 \le \pi$ , its magnitude response  $D(\omega)$  should satisfy the following conditions

 $D(\omega_0) = \omega_0,$ 

and

$$\frac{d^{i}D(\omega)}{d\omega^{i}}\bigg|_{\omega=\omega_{0}} = \begin{cases} 1, & i=1, \\ 0, & 2\leq i\leq L, \end{cases}$$
(9)

(8)

where *L* is a suitable number less than the length of the filter, and depends on the value of  $\omega_0$  at which the MAXLIN constraints are applied.

Magnitude response of a type IV (even number of oddsymmetric coefficients) FIR DD can be expressed as:

$$D(\omega) = 2\sum_{n=1}^{N} d_{n-1/2} \sin(n-1/2)\omega , \qquad (10)$$

where  $d_{n-1/2}$ ,  $-N < n \le N$ , are the IR coefficients.

Applying the MAXLIN constraints of Eqs. (8-9) to Eq. (10), at middle of the frequency band  $\omega = \pi/2$ , we get a set of equations similar to Eq. (5), which can be solved for **d** into closed form as given below [12]:

$$d_{n-1/2} = \begin{cases} \frac{\pi(2N-1)!!S_{N,n}}{2^N\sqrt{2}(2n-1)(N-n)!!(N+n-2)!!} \\ N-n = even \\ \frac{\pi(2N-1)!!S_{N,n}}{2^N\sqrt{2}(2n-1)(N-n-1)!!(N+n-1)!!} \\ N-n = odd, \\ 1 \le n \le N, \end{cases}$$
(11)

where

$$S_{N,n} = 1 - \frac{4}{\pi} \sum_{\substack{1 \le i \le N \\ i \ne n}} \frac{(-1)^{i-1}}{2i-1} \,. \tag{12}$$

It can be noted from Eq. (7) that magnitude response of a DD can be transformed to a halfband lowpass response by

- a) taking its derivative with a suitable scaling factor, and
- b) shrinking its width by half, *i.e.*, fitting it in the region  $0 \le \omega \le \pi$ .

In time domain, these transformations can be carried out on IR coefficients of a DD given by Eqs. (11-12) as [13]:

- i. Multiply  $d_{\pm(n-1/2)}$  by  $\pm(2n-1)/4$ , for  $1 \le n \le N$ .
- ii. Insert a zero between every two consecutive coefficients.
- iii. Set the middle coefficient to  $\frac{1}{2}$ .

The resultant formulas for IR coefficients of halfband lowpass DF can be written as:

$$h_{0} = 1/2$$

$$h_{1} = \begin{cases} \frac{\pi(2N-1)!!S_{N,n}}{2^{N+2}\sqrt{2}(N-n)!!(N+n-2)!!} \\ N-n = even \\ \frac{\pi(2N-1)!!S_{N,n}}{2^{N+2}\sqrt{2}(N-n-1)!!(N+n-1)!!} \\ N-n = odd \\ 1 \le n \le N \end{cases}$$

$$h_{+2n} = 0, \ 1 \le n < N$$

$$(13)$$

where  $S_{N,n}$  is given by Eq. (12).



Fig. 1: Magnitude response of a presented halfband lowpass filter (solid line) designed with Eq. (6) for length of 55 (N = 14) compared with the traditional design of the same length.



Fig. 2: Magnitude response of a presented halfband lowpass filter (solid line) designed with Eq. (13) for length of 55 (N = 14) compared with the traditional design of the same length.



**Fig. 3:** Magnitude responses of presented halfband lowpass filters, designed with Eq. (13) (solid line) and Eq. (6) (dashed line), each of length 55 (N = 14).

A filter designed by the above procedure is compared with a traditional halfband lowpass filter in Fig. 2. Another comparison is shown in Fig. 3, with the filter presented earlier and given by Eq. (6). It can be noted from the figures that the transition band of this new filter is almost as narrow as that of the filter given by Eq. (6), and it has quite smooth response at the ends of the frequency band. In fact, the maximum deviation from ideal is just 0.08% for the presented example, and that occurs at the ends of the frequency band, and is negligible for almost any application.

The reason of higher accuracy at the ends of the frequency band for filters designed by Eq. (13) compared to those designed with Eq. (6) can be understood by looking at the magnitude response of a DD given by Eq. (10). It can be noted that although the design was carried out by applying MAXLIN constraints at  $\omega = \pi/2$ , the first constraint (ideal magnitude at certain frequency) is inherently satisfied at  $\omega$ =0 as well. Due to this reason, DDs designed by using Eq. (11) have lesser relative errors at lower end as compared to the higher end of the frequency band. Therefore, the derivatives of their frequency responses at  $\omega = 0$  and  $\omega = 2\pi$ , which become the magnitudes of halfband filters at  $\omega = 0$  and  $\omega = \pi$  as a result of the above-mentioned transformations, have smaller errors as compared to the designs of Eq. (6) obtained by directly applying MAXFLAT constraints at  $\omega =$  $\pi/4.$ 

Using the coefficients given by Eq. (6) in Eq. (4), it can be shown that for a filter of length 4N-1, the magnitude response and its first N-1 derivatives have the ideal values, i.e., all the *N* MAXFLAT constraints are satisfied. However, a filter of the same length designed Eq. (13) satisfies first N-1 constraints, while the last constraint is not satisfied. The effect on the transition band is however negligible, as can be seen in Fig. 3.

A halfband highpass frequency response can be obtained simply by subtracting a halfband lowpass response from an allpass response. In time domain, it can be done by inverting the sign of all coefficients in Eq. (6) and Eq. (13), except for the central coefficients  $h_0$ , that remain unchanged.

#### C. Implementation Issues

It should be noted that determination of IR coefficients for both of the presented designs given by Eq. (6) and Eq. (13), respectively, involves computation of large factorials. This seems computationally burdensome, however, both Eq. (6) and Eq. (13) can be written in iterative forms, and factorials need to be calculated only once for calculating the first coefficient. All other coefficients can be calculated by simple multiplications.

Similarly,  $S_{N,n}$  in Eq. (12) can be written in an alternate form as

$$S_{N,k} = 1 - \frac{4}{\pi} \sum_{i=1}^{N} \frac{(-1)^{i-1}}{2i-1} + \frac{4}{\pi} \cdot \frac{(-1)^{n-1}}{2n-1}, \qquad (14)$$

in which the summation term is independent of n, and it needs to be calculated only once, and therefore  $S_{N,n}$  can be computed very efficiently.

# III. CONCLUSIONS

Explicit formulas for the impulse response coefficients of two new designs of MAXFLAT FIR halfband low/high pass filters have been obtained. These filters satisfy MAXFLAT constraints at the middle of the pass and stop bands, and have relatively narrow bands as compared to the existing MAXFLAT halfband designs. One of the presented designs, based on direct application of MAXFLAT constraints, is inaccurate at the ends of the frequency band. However, the other design derived from an existing design of maximally linear digital differentiators, is quite accurate in the entire pass and stop bands.

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