A NEW ONE-STEP BAND-LIMITED EXTRAPOLATION PROCEDURE USING EMPIRICAL ORTHOGONAL FUNCTIONS

Jianfeng Weng

School of Information and Electronic Engineering
Zhejiang University of Science and Technology
Hangzhou, China
maple_weng@yahoo.com.cn

ABSTRACT

A one-step band-limited extrapolation procedure is systematically developed under an a priori assumption of bandwidth. The rationale of the proposed scheme is to expand the known signal segment based on a band-limited basis function set and then to generate a set of empirical orthogonal functions (EOF's) adaptively from the sample values of the band-limited function set. Simulation results indicate that, in addition to the attractive adaptive feature, this scheme also appears to guarantee a smooth result for inexact data, thus suggesting the robustness of the proposed procedure.

1. INTRODUCTION

The problem of extrapolating the unknown part of a band-limited signal from its known segment under an a priori bandwidth assumption is a classic problem in spectrum analysis [1-12]. Mathematically, denoting the known segment of a band-limited signal $f(t)$ with an a priori known bandwidth B by $g(t)$

$$g(t) = f(t)\cdot P_\tau(t) = \begin{cases} f(t) & |t| \le \tau \\ 0 & |t| > \tau \end{cases} \tag{1}$$

then to extrapolate the unknown part of $f(t)$ is equivalent to solving for its Fourier transform $F(j\omega)$ from the following integral equation

$$G(j\omega) = \int_{-B}^{B} F(j\nu)\frac{Sin(\omega-\nu)\tau}{\pi(\omega-\nu)}d\nu \tag{2}$$

where $G(j\omega)$ is the Fourier transform of the known segment $g(t)$.

Equation (2) is a Fredholm integral equation of the first kind with a Hermitian kernel, thus ill-posed in nature [13]. As a consequence, small perturbation in $g(t)$ may cause great oscillation in the solution.

During the past thirty years, a good deal band-limited extrapolation procedures have been proposed. Due to the ill-posed nature of the problem, however, the conventional Papoulis-Gerchberg iteration algorithms or the versions using prolate spheroidal wave functions [14, 15] sometimes may not lead to smooth results because of the cumulative arithmetic error or the errors in the data. Same situation can also be seen in some one-step extrapolation algorithm case. As to those algorithms based on the Tikhonov's regularization method, it appears that more efforts are needed to improve the algorithm efficiency, especially the efforts of the choice of regularization parameter.

In what follows, a one-step extrapolation procedure will be presented. By approximating the known signal segment by a finite sum based on the band-limited function set $\left\{\left(\frac{SinBt}{\pi t}\right)^{(k)}\right\}$, a set of empirical orthogonal functions (EOF's) can then be adaptively generated from the sample values of that band-limited basis functions. After the coefficients of the abovementioned expansion have been obtained, the extrapolation of the unknown part of the signal is straightforward. As will be seen, the proposed scheme appears to be robust in the sense that it yields a smooth result for inexact data case. Besides, the computational complexity is relatively low. Therefore, the proposed procedure seems to be applicable to real applications.

2. EXTRAPOLATION SCHEME

Without loss of generality, assume that $f(t) \in L_2$ is real valued. Let us start with the attempt at obtaining the solution approximately using a representation of the known signal segment $f(t)$ ($|t| \le \tau$) as a finite sum:

$$f(t) = \sum_{k=0}^{N-1} x_k \Psi_k(t) \qquad |t| \le \tau \tag{3}$$

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where $x_k's$ $(k = 0,1,\cdots,N-1)$ are real valued coefficients to be determined, $\{\psi_k(t)\}$ is a certain set of band-limited functions.

An initiative choice of $\psi_k(t)$ is the prolate spheroidal wave function and the corresponding coefficients can be obtained by

$$x_k = \frac{1}{\lambda_k}\int_{-\tau}^{\tau} f(t)\psi_k(t)dt \qquad (4)$$

where $\lambda_k's$ are the corresponding eigenvalues and both $\lambda_k's$ and $\psi_k's$ are functions of the product $B\tau$. Apparently, the computational burden of (4) is heavy. Moreover, the accuracy of $x_k's$ would be affected greatly by the errors in the data when the product $B\tau$ is small as $\lambda_k's$ will be small in this case, especially for large k.

The sampling function set $\left\{\frac{Sin(Bt-k\pi)}{Bt-k\pi}\right\}$ is also not a proper choice. In fact, since $Sin(Bt-k\pi) = (-1)^k Sin(Bt)$, (3) can be expressed as

$$f(t) = F_{2(N-1)}(t)Q(t) \qquad (5)$$

where $F_{2(N-1)}(t)$ is a polynomial of degree $2(N-1)$ and

$$Q(t) = \frac{Sin(Bt)}{Bt\prod_{k=1}^{N-1}(1-B^2t^2/k^2\pi^2)} \qquad (6)$$

Equation (6) is arbitrarily close to unity on any finite interval when N is sufficiently large. This implies that the extrapolation is now made by use a time domain polynomial. As a band-limited signal can't be a time-limited signal, the extrapolation error would be wild for large time index values.

Our choice is the function set $\left\{\left(\frac{SinBt}{\pi t}\right)^{(k)}\right\}$. This seems to be a desirable basis function set as the functions $\left(\frac{SinBt}{\pi t}\right)^{(k)}$ $(k = 0,1,\cdots)$ are of exponential type and band-limited. In fact, inserting this function set into (3) and taking Fourier transform on both sides, it can be found that expanding the known signal segment this way is equivalent to approximating the Fourier transform of the band-limited signal $f(t)$ by an $(N-1)th$ degree polynomial $F_{N-1}(j\omega)$. Thus, if the optimum polynomial is obtained, the difference between $F(j\omega)$ and the approximation polynomial $F_{N-1}(j\omega)$ will be minimized.

On the other hand, however, this function set is not good enough as the orthogonality does not hold for the basis functions. Therefore, remedy needs to be adopted to have the basis orthogonalized.

Using the abovementioned function set, the discrete form of (3) now becomes

$$f(t_i) \approx \sum_{k=0}^{N-1} x_k \left(\frac{SinBt}{\pi t}\right)^{(k)}\Bigg|_{t=t_i} \quad (i = 1,2,\cdots,M) \qquad (7)$$

where M is the number of the known data points. In practice, this number is usually great than N. The matrix form of (7) is

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{g} \qquad (8)$$

where $\mathbf{x}$, $\mathbf{g}$ are column vectors of dimension N, M, respectively, $\mathbf{A}$ is an M-by-N matrix whose elements are simply the sample values of the above mentioned band-limited function set:

$$a_{ij} = \left(\frac{SinBt}{\pi t}\right)^{(j-1)}\Bigg|_{t=t_i} \quad (i = 1,2,\cdots,M;\ j = 1,2,\cdots,N) \qquad (9)$$

A set of orthogonal basis functions can be generated using the singular value decomposition:

$$\mathbf{A} = \mathbf{U}\cdot\mathbf{\Sigma}\cdot\mathbf{V}^T \qquad (10)$$

Then we will have

$$a_{ij} = \sum_{k=1}^{N}\sigma_k u_{ik} v_{jk} \quad (i = 1,2,\cdots,M;\ j = 1,2,\cdots,N) \qquad (11)$$

and a new orthogonal basis is thus adaptively generated, or equivalently, the empirical modes of **A** are obtained. In fact, by using the above decomposition, the columns of **A** are projected on another orthogonal basis formed by the columns of **U**.

Note that the choice of basis functions here does not involve the constraint of any particular analytic form and the obtained orthogonal basis varies adaptively with the length of the known interval and the spacing of the sampling points. Therefore, the modes thus obtained are really empirical modes. In other words, the columns of **U** are empirical orthogonal functions (EOF's) while the columns of **V** provide the corresponding principal components of **A**.

After the above manipulation, $x_k's$ $(k = 0,1,\cdots,N-1)$ can be obtained from

$$\mathbf{x} = \mathbf{A}^+\cdot\mathbf{g} \qquad (12)$$

where

$$\mathbf{A}^+ = \mathbf{V}\cdot\mathbf{\Sigma}^+\cdot\mathbf{U}^{\mathbf{T}} \qquad (13)$$

and the elements of $\mathbf{A}^+$ are

$$a_{ij}^{\ +} = \sum_{\sigma_k\neq 0}\frac{v_{ik}u_{jk}}{\sigma_k} \quad (i = 1,2,\cdots,N;\ j = 1,2,\cdots,M) \qquad (14)$$

where σ_k's $(k = 1,2,\cdots,N)$ are the singular values of **A**.

Note that it is not the data **g** that is used to generate the set of orthogonal basis functions. Consequently, the orthogonal basis functions thus generated will have got rid of the influence of the errors in the data while will adapt to the spacing of the sampling points t_i's $(i = 1,2,\cdots,M)$.

Note also that, as the elements of **A** are solely determined by the function set $\left\{\left(\frac{SinBt}{\pi t}\right)^{(k)}\right\}$, which is a set of analytic functions, it appears that the characteristic of the empirical modes thus generated will be insensible to the length of the known signal segment and the spacing of the data points, as long as the requirement of sampling theorem is fulfilled while M is large enough.

On the other hand, however, the smoothness of the solution may still be affected by the errors in the data due to the ill-conditioned nature of the problem. Therefore, the mode number needs to be reduced so as to discard the insignificant modes. This is equivalent to replacing $\mathbf{A}^+$ in (12) by the effective pseudoinverse and therefore can be done by taking the following cut-off rule:

$$\sigma_k^+ = \begin{cases} \dfrac{1}{\sigma_k} & \sigma_k > \rho \\ 0 & \sigma_k < \rho \end{cases} \tag{15}$$

where ρ is some tolerance measure that reflects the errors in the data.

After some mathematical manipulation on $\|\Delta\mathbf{x}\|_2 / \|\mathbf{x}\|_2$, where $\Delta\mathbf{x}$ denotes the error part in the solution $\mathbf{x}$, we found that the following measure is a proper one:

$$\rho \le \sigma_1 \frac{\|\Delta\mathbf{g}\|_2}{\|\mathbf{g}\|_2} \tag{16}$$

where σ_1 is the largest singular value of **A**, $\Delta\mathbf{g}$ denotes the error part in the data. Due to the restriction of paper length, the details of manipulation will not be given here.

By using the above cut-off rule to reduce the mode number, (14) becomes

$$a_{ij}^{\ +} = \sum_{\sigma_k > \rho} \frac{v_{ik} u_{jk}}{\sigma_k} \qquad (i = 1,2,\cdots,N;\ j = 1,2,\cdots,M) \tag{17}$$

When all x_k's $(k = 0,1,\cdots,N-1)$ have been obtained, the signal outside the interval $[-\tau,\tau]$ can be extrapolated using (3) and the corresponding extrapolation error can be measured by

$$\delta = \frac{\|\mathbf{F} - \mathbf{F}_{N-1}\|^2}{\|\mathbf{F}\|^2} \tag{18}$$

3. SIMULATION RESULTS

To verify the effectiveness of the proposed scheme, computer simulation has been conducted for the following two signal pairs:

$$\frac{\sin Bt}{\pi t} \leftrightarrow P_B(\omega) \tag{19}$$

$$\frac{2}{\pi}\left[\frac{Sin(Bt/2)}{t}\right]^2 \leftrightarrow \left(1 - \left|\frac{\omega}{B}\right|\right) P_B(\omega) \tag{20}$$

where the signal pair (19) is purposely selected as the same in [1] for comparison. As to the second signal pair, it is much tougher since its Fourier transform does not have derivatives over the total bandwidth range $[-B,B]$. In fact, our simulation indicates that the algorithms proposed by [1,3] can't achieve satisfactory results for this case, especially when the data are inexact.

As mentioned before, the EOF's are purely derived from the analytic function set $\left\{\left(\frac{SinBt}{\pi t}\right)^{(k)}\right\}$ and therefore the spacing of the sampling points will have little influence on the characteristic of EOF's. To verify this property, both uniform and non-uniform sampling points are adopted under the condition that the requirement of sampling theorem is fulfilled while M maintains large enough.

The elements of the matrix **A**, a_{ij} in (9), can be evaluated either from the truncated Taylor expansion or from the recursive formula of $\left(\frac{SinBt}{\pi t}\right)^{(k)}$.

For the first signal pair (19), simulation is conducted for exact data and $\tau = \frac{\pi}{60B} \sim \frac{\pi}{3B}$, $N=5\sim9$. In all cases, perfect result is achieved, i.e., the approximation polynomial $F_{N-1}(j\omega) = 1$. This result is better than [1]. Although it seems simply because the Fourier transform in this case can be exactly expressed by a polynomial, however, it still verifies the effectiveness of the proposed scheme.

Similar simulation is conducted for the second signal pair (20). Table 1 shows the results for a 6-th degree polynomial approximation for exact data and Fig.1 gives the overlaid plots of $F_6(j\omega)$ shown in Table 1. As can be seen, the resulting five curves are overlapped each other, thus verifying that, nearly identical results for a fixed N can be obtained for different length of the known segment when the data are exact.

In order to examine the effect caused by inexact data, simulation is conducted for signal pair (20) using a 4-th degree polynomial under the following conditions: $\tau = \frac{\pi}{5B}$, $\frac{\pi}{10B}$; $\|\Delta\mathbf{g}\|^2 / \|\mathbf{g}\|^2 = 10^{-3}$, 10^{-4} , 10^{-5} . Both uniform and non-uniform sampling points are used.

Shown in Fig.2 are the overlaid plots of the resulting $F_4(j\omega)$. As can be seen from Fig.2, smooth results are obtained for all cases and the differences between each are

TABLE I. SIGNAL PAIR (20), 6^{TH} DEGREE, $X_1=X_3=X_5=0$

τ	x_0	x_2	x_4	x_6
τ_1	0.91457	-2.30747	2.82084	-1.46705
τ_2	0.91455	-2.30715	2.81990	-1.46637
τ_3	0.91456	-2.30739	2.82061	-1.46689
τ_4	0.91460	-2.30818	2.82298	-1.46862
τ_5	0.91467	-2.30963	2.82735	-1.47183

Note: $\tau_1 = \frac{\pi}{60B}, \tau_2 = \frac{\pi}{30B}, \tau_3 = \frac{\pi}{10B}, \tau_4 = \frac{\pi}{5B}, \tau_5 = \frac{\pi}{3B}$

relatively small. Besides, the associated extrapolation errors measured by (18) are about in the order of 10^{-3} for all cases, thus verifying the effectiveness and suggesting the robustness of the proposed procedure.

4. CONCLUSION

A one-step band-limited extrapolation procedure employing the EOF's is presented. Under an a priori assumption of bandwidth, by expressing the known signal segment as a finite sum of the band-limited function set $\left\{\left(\frac{SinBt}{\pi t}\right)^{(k)}\right\}$, a set of empirical orthogonal functions (EOF's) is then generated from the sample values of the above function set adaptively with the known segment length and the spacing of the sampling points. Computer simulation results verify the adaptive property and the effectiveness of the proposed procedure. It has also been found that, the proposed scheme appears to be robust in the sense that a smooth result is usually guaranteed for inexact data case while the extrapolation error is relatively small. Moreover, due to its one-step extrapolation nature, the involved algorithm is also relatively simple. Therefore, the proposed scheme seems to be applicable to real applications.

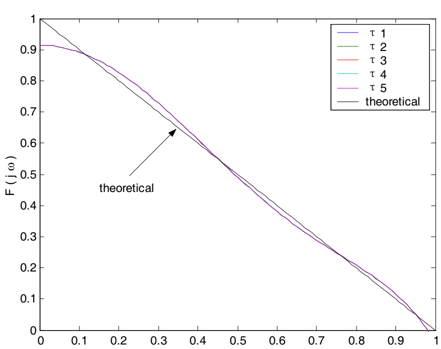

Figure.1 Overlaid Plots of $F_6(j\omega)$ of Signal (20) for Exact Data

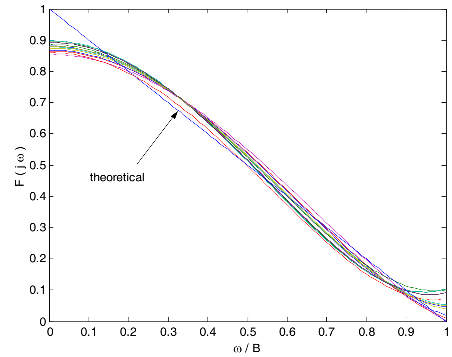

Figure.2 Overlaid Plots of $F_4(j\omega)$ of Signal (20) for Inexact Data

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