A DIRECT METHOD TO GENERATE APPROXIMATIONS OF THE BARANKIN BOUND

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ABSTRACT

The search for an easily computable but tight approximation of the Barankin Bound (BB) is important for the prediction of the Signal-to-Noise Ratio (SNR) value where the Cramer-Rao bound (CRB) becomes unreliable for prediction of Maximum Likelihood Estimators (MLE) variance. In this paper we propose a method for the derivation of a general class of BB approximations which has the advantage of a clear interpretation. This method suggests a new practical BB approximation, whose computational complexity does not exceed that of the CRB but which seems tighter than existing approximations.

1. INTRODUCTION

Minimal performance bounds allow for calculation of the best performance that can be achieved in the Mean Square Error (MSE) sense, when estimating a parameter of a signal corrupted by noise. In the present paper, the parameters being estimated are considered to be deterministic [1] and to be embedded in a noise signal whose parameters are also considered deterministic. Historically the first MSE lower bound for deterministic parameters to be derived was the CRB[2], which has been the most widely used since. Its popularity is largely due to its simplicity of calculation, the fact that in many cases it can be achieved asymptotically (high SNR [3] and/or large number of snapshots [2]) by MLE [1], and last but not least, its noticeable property of being the lowest bound on the MSE of unbiased estimators, since it derives from the *weakest* formulation of unbiasedness at the vicinity of any selected value of the parameters [4][5][6]. This initial characterization of locally unbiased estimators has been improved first by Bhattacharyya's works [1][4] which refined the characterization of local unbiasedness, and significantly generalized by Barankin works [4], who established the general form of the greatest lower bound of any absolute moment of an unbiased estimator. In the particular case of MSE, his work allows the derivation of the highest lower bound on MSE (BB) since it takes into account the *strongest* formulation of unbiasedness, that is to say unbiasedness over an interval of parameter values including the selected value. Unfortunately the BB is generally incomputable [7]. Therefore, since then, numerous works [5][6][8][9] have been devoted to deriving computable approximations of the BB. These works have shown that in non-linear estimation problems three distinct regions of operation can be observed. In the asymptotic region, the MSE is small and, in many cases, close to the Small-Error bounds (CRB). In the *a priori* performance region where the number of independent snapshots and/or the SNR are very low, the observations provide little information and the MSE is close to that obtained from the prior knowledge about the problem. Between these two extremes, there is an additional ambiguity region, also called the transition region. In this region, the MSE of MLEs usually deteriorates rapidly with respect to Small-Error bounds and exhibits a threshold behavior corresponding to a "performance breakdown" highlighted by Large-Error bounds (BB)[6][9][10].

As a result, the search for an easily computable but tight approximation of the BB is still a subject worth investigation. Indeed, the accurate knowledge of the BB should allow a better prediction of the SNR value at the start of the transition region and avoid misleading conclusions - too optimistic - being drawn from the computation of the CRB at low SNR. As a contribution to this research effort, we present in this paper a formalism (see §3) that allows not only the derivation of a general class of BB approximations but also gives a clear interpretation of these approximations (including all previously derived bounds). This formalism suggests a new practical approximation of the BB, whose computational complexity does not exceed that of the CRB, but seems tighter than existing approximations (see §4).

2. OVERVIEW OF BARANKIN BOUND LITERATURE

For the sake of simplicity we will focus on the estimation of a single real function $g(\theta)$ of a single unknown real deterministic parameter θ. In the following, unless otherwise stated, $\mathbf{x}$ denotes the random observations vector, Ω the observation space, and $f_\theta(\mathbf{x})$ the probability density function (p.d.f.) of observations depending on $\theta \in \Theta$, where Θ denotes the parameter space. Let $\mathbb{F}_\Omega$ be the real vector space of square integrable functions over Ω.

2.1. On lower bounds and norm minimization

A fundamental property of the MSE of a particular estimator $\widehat{g(\theta_0)}(\mathbf{x}) \in \mathbb{F}_\Omega$ of $g(\theta_0)$, where θ_0 is a selected value of the parameter θ, is that it is a norm associated with a particular scalar product $\langle \mid \rangle_\theta$:

$$MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] = \left\|\widehat{g(\theta_0)}(\mathbf{x}) - g(\theta_0)\right\|^2_{\theta_0}$$
$$\langle g(\mathbf{x}) \mid h(\mathbf{x})\rangle_{\theta_0} = E_{\theta_0}\left[g(\mathbf{x})\, h(\mathbf{x})\right] = \int g(\mathbf{x})\, h(\mathbf{x})\, f_{\theta_0}(\mathbf{x})\, d\mathbf{x}.$$

In the search for a lower bound on the MSE, this property allows the use of two equivalent fundamental results: the generalisation of the Cauchy-Schwartz inequality to Gram matrices (generally referred to as the "covariance inequality" [9]) and the minimization of a norm under linear constraints [7] [8]. Nevertheless, we shall prefer the "norm minimization" form as its use provides a better understanding of the hypotheses associated with the different lower bounds on the MSE. Then, let $\mathbb{U}$ be a Euclidean vector space of any dimension (finite or infinite) on the body of real numbers $\mathbb{R}$ which has a scalar product $\langle \mid \rangle$. Let $(\mathbf{c}_1, \ldots, \mathbf{c}_K)$ be a free family of K vectors of $\mathbb{U}$ and $\mathbf{v} = (v_1, \ldots, v_K)^T$ a vector of $\mathbb{R}^K$. The problem of the minimization of $\|\mathbf{u}\|^2$ under the K linear constraints $\langle \mathbf{u} \mid \mathbf{c}_k \rangle = v_k, k \in [1, K]$ then has the solution [7][8]:

$$\begin{gathered}\min\left\{\|\mathbf{u}\|^2\right\} = \mathbf{v}^T\mathbf{G}^{-1}\mathbf{v} \quad \text{for} \quad \mathbf{u}_{opt} = \sum_{k=1}^{K} \alpha_k \mathbf{c}_k \\ (\alpha_1, \ldots, \alpha_K)^T = \boldsymbol{\alpha} = \mathbf{G}^{-1}\mathbf{v}, \quad \mathbf{G}_{n,k} = \langle \mathbf{u}_k \mid \mathbf{c}_n \rangle\end{gathered} \tag{1}$$

2.2. Application to Barankin Bound derivation

As introduced by Barankin, the ultimate constraint that an unbiased estimator $\widehat{g(\theta_0)}(\mathbf{x})$ of $g(\theta)$ should verify is to be unbiased for all possible values of the unknown parameter:

$$E_\theta\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta), \forall \theta \in \Theta. \tag{2}$$

In this case the problem of interest becomes:

$$\min\left\{MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right]\right\} \text{ under } E_\theta\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta), \tag{3}$$

$\forall \theta \in \Theta$ and corresponds to the search for the locally-best unbiased estimator. This problem can be solved by applying the work of Barankin [4] that has been supported by many other studies [6][7][8] aimed not only at expressing the principal results of Barankin's (mathematical) theory in a form accessible to most engineers, but also at obtaining "computable" lower bounds approximating the BB. In the following we provide a synthesis of these studies, highlighting all the key results (4) (5) (7) (8) (9a-d). If $\widehat{g(\theta_0)}(\mathbf{x})$ is an unbiased estimator of $g(\theta)$ in the Barankin sense (2), then:

$$E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n) = \int \widehat{g(\theta_0)}(\mathbf{x}) f_{\theta_n}(\mathbf{x}) d\mathbf{x}, \forall \theta_n \in \Theta.$$

Consequently, $\forall \mathbf{w} \in \mathbb{R}^N$:

$$E_{\theta_0}\left[\left(\widehat{g(\theta_0)}(\mathbf{x}) - g(\theta_0)\right)\left(\sum_{n=1}^{N} w_n \frac{f_{\theta_n}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}\right)\right] = \sum_{n=1}^{N} w_n \left(g(\theta_n) - g(\theta_0)\right)$$

Therefore, according to (1), the minimization of $MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right]$ under the constraint as above - valid for any subset of test points $\{\theta_n\}_{[1,N]}$ of Θ and $\mathbf{w} \in \mathbb{R}^N$ - implies [4]:

$$MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] \geq \lim_{N\to\infty} \sup_{\mathbf{w},\{\theta_n\}_{[1,N]}} \frac{\left[\sum_{n=1}^{N} w_n \left(g(\theta_n) - g(\theta_0)\right)\right]^2}{E_{\theta_0}\left[\left(\sum_{n=1}^{N} w_n \frac{f_{\theta_n}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}\right)^2\right]} \tag{4}$$

which is the original form of the BB on MSE. A more concise form can be derived by noting that [6]:

$$\frac{\left[\sum_{n=1}^{N} w_n (g(\theta_n) - g(\theta_0))\right]^2}{E_{\theta_0}\left[\left(\sum_{n=1}^{N} w_n \frac{f_{\theta_n}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})}\right)^2\right]} = \frac{\left(\mathbf{w}^T\boldsymbol{\Delta}\mathbf{g}\right)^2}{\mathbf{w}^T\mathbf{R}\mathbf{w}} \leq \boldsymbol{\Delta}\mathbf{g}^T\mathbf{R}^{-1}\boldsymbol{\Delta}\mathbf{g} \tag{5}$$

$$R_{n,m} = \int \frac{f_{\theta_n}(\mathbf{x}) f_{\theta_m}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})} d\mathbf{x}, \quad \Delta g_n = g(\theta_n) - g(\theta_0)$$

- since $\frac{\left(\mathbf{w}^T\boldsymbol{\Delta}\mathbf{g}\right)^2}{\mathbf{w}^T\mathbf{R}\mathbf{w}} \leq \boldsymbol{\Delta}\mathbf{g}^T\mathbf{R}^{-1}\boldsymbol{\Delta}\mathbf{g}$ and reaches its maximum value for $\mathbf{w} = \lambda\mathbf{R}^{-1}\boldsymbol{\Delta}\mathbf{g}$ - which leads to the "reduced" form of the BB [6]:

$$MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] \geq \lim_{N\to\infty} \sup_{\{\theta_n\}_{[1,N]}} \left\{\boldsymbol{\Delta}\mathbf{g}^T\mathbf{R}^{-1}\boldsymbol{\Delta}\mathbf{g}\right\} \tag{6}$$

It is then worth noting that (5) is also the solution of:

$$\min\left\{MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right]\right\} \text{ under } E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n) \tag{7}$$

where $\{\theta_n\}_{[1,N]} \in \Theta$, which corresponds to the fact that the greatest lower bound on the MSE for a finite number of test points $\{\theta_n\}_{[1,N]}$ is obtained by simply expressing the "unbiased" constraint at the test points [6]. Consequently the unbiased and locally best estimator $\widehat{g(\theta_0)}_{opt}$ satisfies (5)[8]:

$$\lim_{N\longrightarrow\infty}\left|\begin{array}{c}\mathbf{R}\left(\frac{\mathbf{w}}{\lambda}\right) = \boldsymbol{\Delta}\mathbf{g} \\ \widehat{g(\theta_0)}_{opt}(\mathbf{x}) - g(\theta_0) = \sum_{n=1}^{N} \frac{w_n}{\lambda} \frac{f_{\theta_n}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})} \\ MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] \geq \boldsymbol{\Delta}\mathbf{g}^T\mathbf{R}^{-1}\boldsymbol{\Delta}\mathbf{g} = \boldsymbol{\Delta}\mathbf{g}^T\left(\frac{\mathbf{w}}{\lambda}\right)\end{array}\right. \tag{8}$$

that leads to, defining $\frac{1}{\lambda} = d\theta = \theta_{n+1} - \theta_n$, [7]:

$$\int K(\theta, \theta') w(\theta') d\theta' = g(\theta) - g(\theta_0) \tag{9a}$$

$$K(\theta, \theta') = \int \frac{f_\theta(\mathbf{x}) f_{\theta'}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})} d\mathbf{x} \tag{9b}$$

$$\widehat{g(\theta_0)}_{opt}(\mathbf{x}) - g(\theta_0) = \int \frac{f_\theta(\mathbf{x})}{f_{\theta_0}(\mathbf{x})} w(\theta) d\theta \tag{9c}$$

$$MSE_{\theta_0}\left[\widehat{g(\theta_0)}\right] \geq \int \left(g(\theta) - g(\theta_0)\right) w(\theta) d\theta \tag{9d}$$

Unfortunately, in most practical cases, it is impossible to find either the limit of (6) or an analytical solution of (9a) to obtain an explicit form of $\widehat{g(\theta_0)}_{opt}$ and of the lower bound on the MSE, which somewhat limits its interest. Therefore, the search for an easily computable but tight approximation of the BB is a subject of great theoretical and practical importance.

3. APPROXIMATING THE BARANKIN BOUND

So far, all previous works dedicated to assessing the true behaviour of the BB at low SNR (Large-Error bounds) [6]-[9] can be reduced to the exploitation of the norm minimisation lemma (1) associated with a basic discretisation (7) of Barankin unbiasedness definition (2). Such a basic discretisation is sub-optimal in the scope of BB approximation tightness. Indeed, there is a set $\mathcal{H}$ of numerous functions $h(\theta)$ of various behaviour that take the same values for a given set of test points $\left(h(\theta_n) = g(\theta_n), \{\theta_n\}_{[1,N]}\right)$. Therefore, the lower bound provided by such a discretisation (7) may not be a tight BB approximation since it is a lower bound for the whole set of functions $\mathcal{H}$, except when the number of test points θ_n tends to infinity as $\mathcal{H}$ tends to reduce to $g(\theta)$ only. As a consequence, in order to reduce the set $\mathcal{H}$ and thereby increase the tightness, it seems intuitively more efficient to resort to constraints that are more discriminating, such as l^{th} order derivative constraints. This leads to a straightforward, but to our best knowledge, novel method of approximating the BB.

The development of this method requires that both $f_\theta(\mathbf{x})$ and $g(\theta)$ can be approximated by piecewise series expansions of order L_n, that is to say the parameter space Θ can be partitioned in N real sub-intervals I_n over which - $\theta_n + d\theta \in I_n$ -:

$$\begin{aligned} g(\theta_n + d\theta) = g(\theta_n) \quad &+ \sum_{l=1}^{L_n} \frac{\partial^l g(\theta_n)}{\partial^l \theta} \frac{d\theta^l}{l!} + o\left(d\theta^l\right) \\ f_{\theta_n + d\theta}(\mathbf{x}) = f_{\theta_n}(\mathbf{x}) \; &+ \sum_{l=1}^{L_n} \frac{\partial^l f_{\theta_n}(\mathbf{x})}{\partial^l \theta} \frac{d\theta^l}{l!} + o_{\mathbf{x}}\left(d\theta^l\right) \end{aligned}$$

and that the integrals $\int \left(\frac{\partial^l f_\theta(\mathbf{x})}{\partial^l \theta}\right)^2 \frac{1}{f_\theta(\mathbf{x})} d\mathbf{x}$ converge and define piecewise continuous functions of θ on Θ, for all $\theta \in \Theta$, to allow order of integration and differentiation interchange [9]. Then, some straightforward calculations show that, on every sub-interval I_n, a possible general discretisation of Barankin unbiasedness definition (2) is:

$$E_{\theta_n + d\theta}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n + d\theta) + o\left(d\theta^{L_n}\right) \tag{10}$$

provided the $L_n + 1$ linear constraints are verified:

$$\int \widehat{g(\theta_0)}(\mathbf{x}) \frac{\partial^l f_{\theta_n}(\mathbf{x})}{\partial^l \theta} d\mathbf{x} = \frac{\partial^l g(\theta_n)}{\partial^l \theta}, \quad l \in [0, L_n]$$

or equivalently:

$$E_{\theta_0}\left[\left(\widehat{g(\theta_0)}(\mathbf{x}) - g(\theta_0)\right) \frac{\frac{\partial^l f_{\theta_n}(\mathbf{x})}{\partial^l \theta}}{f_{\theta_0}(\mathbf{x})}\right] = \left[\frac{\partial^l (g(\theta) - g(\theta_0))}{\partial^l \theta}\right]_{\theta_n} \tag{11}$$

Thus, the set of $\sum_{n=1}^{N} (L_n + 1)$ constraints (11) deriving from the N piecewise discretisation of (2) defines a given approximation of the BB denoted by $\widehat{\mathrm{BB}}_{L_1,...,L_N}^{I_1,...,I_N}$ (1):

$$\widehat{\mathrm{BB}}_{L_1,...,L_N}^{I_1,...,I_N} = \mathbf{v}^T \mathbf{G}^{-1} \mathbf{v}$$

$$\begin{gathered} \mathbf{v} = \left[\mathbf{v}_1^T, ..., \mathbf{v}_N^T\right]^T, \; \mathbf{G} = E_{\theta_0}\left[\mathbf{c}\mathbf{c}^T\right] \\ \mathbf{v}_n = \left[g(\theta_n) - g(\theta_0), \frac{\partial g(\theta_n)}{\partial \theta}, ..., \frac{\partial^{L_n} g(\theta_n)}{\partial^{L_n} \theta}\right] \\ \mathbf{c} = \left[\mathbf{c}_1^T, ..., \mathbf{c}_N^T\right]^T, \; \mathbf{c}_n = \left[f_{\theta_n}(\mathbf{x}), \frac{\partial f_{\theta_n}(\mathbf{x})}{\partial \theta}, ..., \frac{\partial^{L_n} f_{\theta_n}(\mathbf{x})}{\partial^{L_n} \theta}\right] \end{gathered} \tag{12}$$

Moreover, if $\min\{L_1, ..., L_N\}$ tends to infinity, a straightforward exercise in mean square convergence establishes that $\widehat{\mathrm{BB}}_{L_1,...,L_N}^{I_1,...,I_N}$ converges in mean-square to the BB. An immediate generalisation of expression (12) consists of taking its supremum over existing degrees of freedom (sub-interval definitions and series expansion orders). Lastly, it is worth noting that the proposed formalism allows exploration of the unbiasedness assumption from its *weakest* to its *strongest* formulation.

3.1. A different look at existing BB approximations

Designating the BB approximations as:

- N-piecewise BB approximation of homogeneous order L, if on all sub-intervals I_n the series expansions are of the same order L,
- N-piecewise BB approximation of heterogeneous orders $\{L_1, ..., L_N\}$, if otherwise,

we can provide a new look at previously derived MSE lower bounds:

- the CRB [2] is a 1-piecewise BB approximation of homogeneous order 1, since the constraints are:

$$E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x}) \frac{\frac{\partial f_{\theta_0}(\mathbf{x})}{\partial \theta}}{f_{\theta_0}(\mathbf{x})}\right] = \frac{\partial g(\theta_0)}{\partial \theta}$$

- the Bhattacharyya bound [1] of order L is a 1-piecewise BB approximation of homogeneous order L, since the constraints are:

$$E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), \; E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x}) \frac{\frac{\partial^l f_{\theta_0}(\mathbf{x})}{\partial^l \theta}}{f_{\theta_0}(\mathbf{x})}\right] = \frac{\partial^l g(\theta_0)}{\partial^l \theta}$$

- the Hammersley-Chapman-Robbins bound (HCRB) [5] is the supremum of a 2-piecewise BB approximation of homogeneous order 0, over a set of constraints of type:

$$E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), \; E_{\theta_1}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_1)$$

- the McAulay-Seidman bound (MSB_N) [6] with N test points is an $N + 1$-piecewise BB approximation of homogeneous order 0, since the constraints are:

$$E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), \; E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n)$$

- the Hybrid Barankin-Bhattacharyya bound ($\mathrm{HBB}_{L,N}$) [9] is an $N{+}1$-piecewise BB approximation of heterogeneous order $\{L, 0, ..., 0\}$, since the constraints are:

$$\left\{ \begin{array}{c} E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), \; E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x}) \frac{\frac{\partial^l f_{\theta_0}(\mathbf{x})}{\partial^l \theta}}{f_{\theta_0}(\mathbf{x})}\right] = \frac{\partial^l g(\theta_0)}{\partial^l \theta} \\ E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n) \end{array} \right.$$

3.2. A new practical BB approximation

The formalism introduced suggests a very straightforward practical BB approximation - denoted $\widehat{\mathrm{BB}}_1^N$ in the following -: the $N{+}1$-piecewise BB approximation of homogeneous order 1 characterized by the set of constraints:

$$\left\{ \begin{array}{l} E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_0), \; E_{\theta_0}\left[\widehat{g(\theta_0)}(\mathbf{x}) \frac{\frac{\partial f_{\theta_0}(\mathbf{x})}{\partial \theta}}{f_{\theta_0}(\mathbf{x})}\right] = \frac{\partial g(\theta_0)}{\partial \theta} \\ E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x})\right] = g(\theta_n), \; E_{\theta_n}\left[\widehat{g(\theta_0)}(\mathbf{x}) \frac{\frac{\partial f_{\theta_n}(\mathbf{x})}{\partial \theta}}{f_{\theta_n}(\mathbf{x})}\right] = \frac{\partial g(\theta_n)}{\partial \theta} \end{array} \right.$$

Indeed it appears to be the generalization of the CRB when the parameter space is partitioned in more than one sub-interval, as well as the generalization of the usual BB approximation used in the open literature, i.e. the McAulay-Seidman form of the BB[6]. Therefore its computational complexity doesn't exceed that of these two bounds. In the next section we will show using a standard spectral analysis problem - single tone parameter estimation - that this new BB approximation is tighter than existing ones.

4. SINGLE TONE THRESHOLD ANALYSIS

Let the complex observation vector $\mathbf{x}$ be modelled by:

$$\begin{gathered} \mathbf{x} = s\boldsymbol{\psi}(\theta_0) + \mathbf{n} \\ \boldsymbol{\psi}(\theta_0) = \left[1, e^{j2\pi\theta_0}, ..., e^{j(K-1)2\pi\theta_0}\right]^T, \quad \theta_0 \in]-0.5, 0.5[\end{gathered}$$

where θ_0 is the unknown parameter to estimate, s^2 is the SNR ($s > 0$) and $\mathbf{n}$ is a complex circular Gaussian noise, with zero mean and covariance matrix $\mathbf{C_n} = \mathbf{Id}$. Therefore the p.d.f. of the observations is given by:

$$f_{\theta_0}(\mathbf{x}) = \frac{e^{-\left\|\mathbf{x}-s\boldsymbol{\psi}(\boldsymbol{\theta}_0)\right\|^2}}{\pi^K}$$

For any set of $N+1$ test points $\{\theta_n\}_{[1,N+1]}$, among the existing lower bounds, only the MSB_N and the $\mathrm{HBB}_{1,N}$ are of a complexity comparable with $\widehat{\mathrm{BB}}_1^N$. Nevertheless, we also include in the comparison the CRB and the HCRB as they are the simplest representative of respectively Small Errors bounds and Large Errors bounds. All mentioned lower bounds can be computed from the components of $\widehat{\mathrm{BB}}_1^N$ - (12) with rearrangement -:

$$\mathbf{v} = \left[\boldsymbol{\Delta}\mathbf{g}^T, \left(\ldots, \frac{\partial g(\theta_n)}{\partial\theta}, \ldots\right)^T\right]^T, \ \mathbf{G} = \begin{bmatrix} \mathbf{R} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{F} \end{bmatrix}.$$

Noting that:

$$\begin{aligned} \frac{f_{\theta_n}(\mathbf{x}) f_{\theta_l}(\mathbf{x})}{f_{\theta_0}(\mathbf{x})} &= e^{2s^2 \operatorname{Re}\{[\boldsymbol{\psi}(\theta_n)-\boldsymbol{\psi}(\theta_0)]^H[\boldsymbol{\psi}(\theta_l)-\boldsymbol{\psi}(\theta_0)]\}} f(\mathbf{x}) \\ f(\mathbf{x}) &= \frac{e^{-\|\mathbf{x}-s[\boldsymbol{\psi}(\theta_n)+\boldsymbol{\psi}(\boldsymbol{\theta}_1)-\boldsymbol{\psi}(\theta_0)]\|^2}}{\pi^K}, \end{aligned}$$

$\mathbf{R}$, $\mathbf{C}$, $\mathbf{F}$ matrices are given by:

$$\begin{aligned} &\mathbf{R}_{n,l} = e^{2s^2 \operatorname{Re}\{[\boldsymbol{\psi}(\theta_n)-\boldsymbol{\psi}(\theta_0)]^H[\boldsymbol{\psi}(\theta_l)-\boldsymbol{\psi}(\theta_0)]\}} \\ &\mathbf{C}_{n,l} = 2s^2\left(\mathbf{R}_{n,l}\right) \operatorname{Re}\left\{\left[\boldsymbol{\psi}(\theta_n) - \boldsymbol{\psi}(\theta_0)\right]^H \frac{\partial \boldsymbol{\psi}(\theta_l)}{\partial\theta}\right\} \\ &\mathbf{F}_{n,l} = 2s^2\left(\mathbf{R}_{n,l}\right) \\ &\begin{bmatrix} \operatorname{Re}\left\{\frac{\partial\boldsymbol{\psi}(\theta_l)^H}{\partial\theta} E\left[\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_l)\right)\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_n)\right)^T\right] \frac{\partial\boldsymbol{\psi}(\theta_n)^*}{\partial\theta}\right\} + \\ \operatorname{Re}\left\{\frac{\partial\boldsymbol{\psi}(\theta_l)^H}{\partial\theta} E\left[\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_l)\right)\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_n)\right)^H\right] \frac{\partial\boldsymbol{\psi}(\theta_n)}{\partial\theta}\right\} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} E[\mathbf{x}] &= s\left[\boldsymbol{\psi}(\theta_n) + \boldsymbol{\psi}(\theta_l) - \boldsymbol{\psi}(\theta_0)\right] \\ E\left[\mathbf{x}\mathbf{x}^T\right] &= E[\mathbf{x}]\, E[\mathbf{x}]^T \\ E\left[\mathbf{x}\mathbf{x}^H\right] &= \mathbf{I}d + E[\mathbf{x}]\, E[\mathbf{x}]^H \\ E\left[\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_l)\right)\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_n)\right)^T\right] &= E\left[\mathbf{x}\mathbf{x}^T\right] - sE[\mathbf{x}]\,\boldsymbol{\psi}(\theta_n)^T \\ &\quad -s\boldsymbol{\psi}(\theta_l)\, E[\mathbf{x}]^T + s^2\boldsymbol{\psi}(\theta_l)\,\boldsymbol{\psi}(\theta_n)^T \\ E\left[\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_l)\right)\left(\mathbf{x}-s\boldsymbol{\psi}(\theta_n)\right)^H\right] &= E\left[\mathbf{x}\mathbf{x}^H\right] - sE[\mathbf{x}]\,\boldsymbol{\psi}(\theta_n)^H \\ &\quad -s\boldsymbol{\psi}(\theta_l)\, E[\mathbf{x}]^H + s^2\boldsymbol{\psi}(\theta_l)\,\boldsymbol{\psi}(\theta_n)^H \end{aligned}$$

where $\mathbf{x}^T, \mathbf{x}^*, \mathbf{x}^H$ denotes respectively the transpose, the conjugate, the transpose-conjugate of $\mathbf{x}$.

We consider the reference estimation case where $\theta_0 = 0$. For the sake of fair comparison with the HCRB, the MSB_N, $\mathrm{HBB}_{1,N}$, $\widehat{\mathrm{BB}}_1^N$ are computed as supremum over the possible values of $\{\theta_n\}_{[1,N+1]}$. For the sake of simplicity $\{\theta_n\}_{[1,N+1]} = \{0, d\theta, -d\theta\}$. Figure (1) shows the evolution of the various bounds as a function of SNR in the case of $K = 10$ samples. The variance of the MLE is also shown in order to compare the threshold behaviour of the bounds. As intuitively expected, the proposed $\widehat{\mathrm{BB}}_1^N$ bound results in a tighter BB approximation than the other bounds and allows a better prediction of the SNR threshold value.

Additionally, the present results suggest that the true value of the BB may be significantly underestimated by existing approximations, questioning previously drawn conclusions on MLE variance prediction by Deterministic Large Error Bounds.

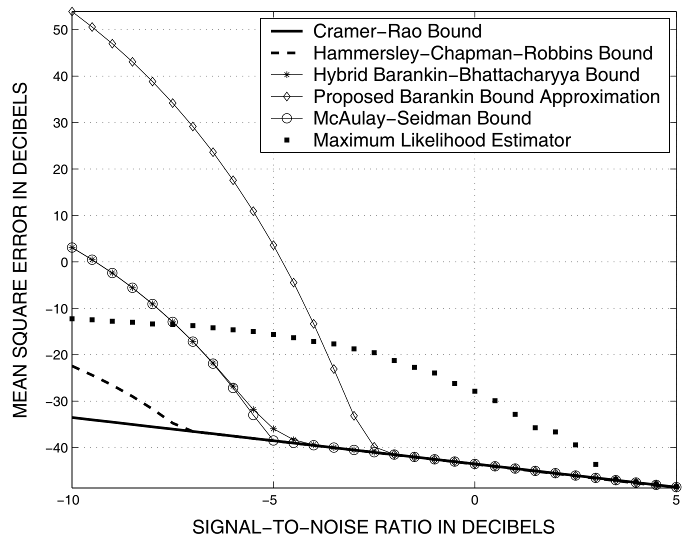

Fig. 1. Comparison of MSE lower bounds versus SNR

5. CONCLUSION

We have proposed a formalism that allows the derivation of a general class of BB approximations and more particularly of a new promising practical approximation, since its computational complexity does not exceed that of the CRB, but seems closer to the BB than existing approximations.

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