A STATISTICAL TEST FOR IMPROPRIETY OF COMPLEX RANDOM SIGNALS

Peter J. Schreier[*], *Louis L. Scharf*[†], *Alfred Hanssen*[‡]

[*] School of Electrical Engineering and Computer Science, University of Newcastle, Australia
email: Peter.Schreier@newcastle.edu.au
[†] Dept. of Electrical and Computer Engineering, Colorado State University, USA
email: scharf@engr.colostate.edu
[‡] Dept. of Physics, Universitetet i Tromsø, Norway
email: alfred@phys.uit.no

ABSTRACT

A complex random vector is called improper if it is correlated with its complex conjugate. In this paper, we present a generalized likelihood ratio test (GLRT) for impropriety. This test is compelling because it displays the right invariances: The proposed GLR is invariant to linear transformations on the data, including rotation and scaling, just as propriety is preserved by linear transformations. Because canonical correlations make up a complete, or maximal, set of invariants for the Hermitian and complementary covariance matrices under linear transformations, the GLR can be shown to be a function of the squared canonical correlations between the data and its complex conjugate. This validates our intuition that the internal coordinate system should not matter for this hypothesis test.

1. INTRODUCTION

In recent years, there has been an increased awareness that the second-order statistics of a complex random vector $\mathbf{s}$ are characterized by *two* covariance matrices (see, e.g., [1] – [5]): the Hermitian, positive semidefinite covariance matrix $\mathbf{\Gamma} = E\left\{(\mathbf{s}-E\mathbf{s})(\mathbf{s}-E\mathbf{s})^H\right\}$ and the complex symmetric, complementary covariance matrix $\mathbf{C} = E\left\{(\mathbf{s}-E\mathbf{s})(\mathbf{s}-E\mathbf{s})^T\right\}$. In the past, it was often assumed that the complementary covariance vanishes, $\mathbf{C} = \mathbf{0}$, a case that is referred to as *proper*, *circular*, or *circularly symmetric*. Recently, it has been shown that there are a number of situations in communications where this is not the case, and taking the information contained in the complementary covariance into account can lead to significant performance gains (e.g., [3, 4]). In order to access the information in the complementary covariance $\mathbf{C}$ if $\mathbf{s}$ is improper, *widely linear* (WL) or conjugate-linear transformations are required. WL transformations depend linearly on the complex vector $\mathbf{s}$ and its complex conjugate $\mathbf{s}^*$ [6], which may be expressed as $\mathbf{A}\mathbf{s}+\mathbf{B}\mathbf{s}^*$.

In many applications, the complementary covariance matrix must be estimated from the data available. Such an estimate $\widehat{\mathbf{C}}$ will in general be nonzero even if the source is in fact proper. The question that arises is how we can classify a problem as proper or improper. To answer this question, we propose a hypothesis test for impropriety based on a generalized likelihood ratio (GLR). In a GLR, the unknown parameters ($\mathbf{\Gamma}$ and $\mathbf{C}$ in our case) are replaced by maximum likelihood estimates [7]. This procedure is not generally optimal in the sense of Neyman-Pearson, but it is widely used in practice because of its reliable performance. When the GLR displays compelling invariances, then it is valued even if it cannot be shown to be uniformly most powerful. We demonstrate that the GLR for our problem indeed satisfies an intuitively appealing invariance to linear transformations on the data, which include rotation and scaling.

2. THE GLRT DECISION RULE

Let $\mathbf{s} \in \mathbb{C}^N$ be a complex Gaussian random vector, and let $\boldsymbol{\sigma} = [\mathbf{s}^T, \mathbf{s}^H]^T$ denote an augmented vector formed by stacking $\mathbf{s}$ on top of its conjugate $\mathbf{s}^*$. In general, the probability density function (pdf) of potentially improper $\mathbf{s}$ is given by [1, 2]

$$p(\boldsymbol{\sigma}) = \pi^{-N}(\det \mathbf{R})^{-1/2}\exp\left\{-\frac{1}{2}(\boldsymbol{\sigma}-\boldsymbol{\mu})^H\mathbf{R}^{-1}(\boldsymbol{\sigma}-\boldsymbol{\mu})\right\}.$$

In this equation, $\boldsymbol{\mu} = E\boldsymbol{\sigma}$ is the augmented mean vector, and

$$\mathbf{R} = E(\boldsymbol{\sigma}-\boldsymbol{\mu})(\boldsymbol{\sigma}-\boldsymbol{\mu})^H = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{C} \\ \mathbf{C}^* & \mathbf{\Gamma}^* \end{bmatrix} \in \mathbb{C}^{2N\times 2N}$$

is the augmented covariance matrix of $\mathbf{s}$ [5]. It contains the Hermitian covariance matrix

$$\mathbf{\Gamma} = E\left\{(\mathbf{s}-E\mathbf{s})(\mathbf{s}-E\mathbf{s})^H\right\}$$

and the complementary covariance matrix

$$\mathbf{C} = E\left\{(\mathbf{s}-E\mathbf{s})(\mathbf{s}-E\mathbf{s})^T\right\}.$$

If $\mathbf{C} = \mathbf{0}$, then $\mathbf{\Gamma}$ completely characterizes the second order properties of $\mathbf{s}$, and $\mathbf{s}$ is called *proper*. Proper random vectors are sometimes also referred to as *circularly symmetric* or *circular*, which stems from the fact that the second order statistics of a proper random vector are invariant to circular rotation: If $\mathbf{s}$ is proper, then $\mathbf{s}$ and $e^{j\alpha}\mathbf{s}$ have the same second order statistics for all real α.

Now consider M independent and identically distributed (i.i.d.) random samples $\mathbf{S} = [\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_M]$ taken from the Gaussian distribution with augmented mean $\boldsymbol{\mu}$ and augmented covariance $\mathbf{R}$. Let $\boldsymbol{\Sigma} = [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_M]$ denote the augmented sample matrix, where $\boldsymbol{\sigma}_m = [\mathbf{s}_m^T, \mathbf{s}_m^H]^T$ is the augmented sample vector formed by stacking $\mathbf{s}_m$ on top of its conjugate $\mathbf{s}_m^*$. Then, the joint pdf of these samples is given by

$$\begin{aligned} p(\boldsymbol{\Sigma}) &= \pi^{-MN}(\det \mathbf{R})^{-M/2} \times \\ &\quad \times \exp\left\{-\frac{1}{2}\sum_{m=1}^{M}(\boldsymbol{\sigma}_m - \boldsymbol{\mu})^H \mathbf{R}^{-1}(\boldsymbol{\sigma}_m - \boldsymbol{\mu})\right\} \\ &= \pi^{-MN}(\det \mathbf{R})^{-M/2} \exp\left\{-\frac{M}{2}\operatorname{tr}(\mathbf{R}^{-1}\widehat{\mathbf{R}})\right\}. \end{aligned}$$

In this equation, tr denotes the trace of a matrix and $\widehat{\mathbf{R}}$ is the sample augmented covariance matrix

$$\widehat{\mathbf{R}} = \frac{1}{M}\sum_{m=1}^{M}(\boldsymbol{\sigma}_m - \widehat{\boldsymbol{\mu}})(\boldsymbol{\sigma}_m - \widehat{\boldsymbol{\mu}})^H = \begin{bmatrix} \widehat{\boldsymbol{\Gamma}} & \widehat{\mathbf{C}} \\ \widehat{\mathbf{C}}^* & \widehat{\boldsymbol{\Gamma}}^* \end{bmatrix}.$$

This matrix contains the sample Hermitian covariance matrix $\widehat{\boldsymbol{\Gamma}}$ and the sample complementary covariance matrix $\widehat{\mathbf{C}}$. Its definition uses the sample augmented mean vector

$$\widehat{\boldsymbol{\mu}} = \frac{1}{M}\sum_{m=1}^{M}\boldsymbol{\sigma}_m.$$

Our aim is to develop a hypothesis test of

$$\begin{aligned} H_0 &: \mathbf{C} = \mathbf{0},\ \mathbf{s} \text{ is proper} \\ H_1 &: \mathbf{C} \neq \mathbf{0},\ \mathbf{s} \text{ is improper.} \end{aligned}$$

We propose a generalized likelihood ratio test (GLRT). Even though a GLRT is not optimal in the sense of Neyman-Pearson, it is simple to implement and generally provides good performance in practice [7]. The GLRT statistic is

$$\lambda = \frac{\max\limits_{\substack{\mathbf{R} \\ \mathbf{C}=\mathbf{0}}} p(\boldsymbol{\Sigma})}{\max\limits_{\mathbf{R}} p(\boldsymbol{\Sigma})}. \tag{1}$$

This is the ratio of likelihood with $\mathbf{R}$ constrained to have zero off-diagonal blocks, $\mathbf{C} = \mathbf{0}$, to likelihood with $\mathbf{R}$ unconstrained. We are thus testing whether or not $\mathbf{R}$ is block-diagonal.

It is well-known that the unconstrained maximum likelihood (ML) estimate of $\mathbf{R}$ is the sample covariance matrix $\widehat{\mathbf{R}}$, and the ML estimate of $\mathbf{R}$ under the constraint $\mathbf{C} = \mathbf{0}$ is

$$\widehat{\mathbf{R}}_{\mathbf{0}} = \begin{bmatrix} \widehat{\boldsymbol{\Gamma}} & \mathbf{0} \\ \mathbf{0} & \widehat{\boldsymbol{\Gamma}}^* \end{bmatrix}.$$

Hence, the GLR (1) can be expressed as

$$\begin{aligned} \ell = \lambda^{\frac{2}{M}} &= \det\left(\widehat{\mathbf{R}}_{\mathbf{0}}^{-1}\widehat{\mathbf{R}}\right)\left(\exp\left\{-\frac{M}{2}\operatorname{tr}\left(\widehat{\mathbf{R}}_{\mathbf{0}}^{-1}\widehat{\mathbf{R}} - \mathbf{I}\right)\right\}\right)^{\frac{2}{M}} \\ &= \det\begin{bmatrix} \mathbf{I} & \widehat{\boldsymbol{\Gamma}}^{-1}\widehat{\mathbf{C}} \\ \widehat{\boldsymbol{\Gamma}}^{-*}\widehat{\mathbf{C}}^* & \mathbf{I} \end{bmatrix} \\ &= \det(\mathbf{I} - \widehat{\boldsymbol{\Gamma}}^{-1}\widehat{\mathbf{C}}\widehat{\boldsymbol{\Gamma}}^{-*}\widehat{\mathbf{C}}^*). \end{aligned} \tag{2}$$

We may also write it as

$$\ell = \frac{\det \widehat{\mathbf{R}}}{(\det \widehat{\boldsymbol{\Gamma}})^2} \tag{3}$$

$$= \frac{\det(\widehat{\boldsymbol{\Gamma}} - \widehat{\mathbf{C}}\widehat{\boldsymbol{\Gamma}}^{-*}\widehat{\mathbf{C}}^*)}{\det \widehat{\boldsymbol{\Gamma}}}. \tag{4}$$

Equations (2) through (4) are equivalent formulations of this GLR. The actual implementation of this test will rely on (3) since it does not require computation of $\widehat{\boldsymbol{\Gamma}}^{-1}$. However, the expression (4) provides more insight: It expresses the GLR as the ratio of the determinant of the Schur complement of $\widehat{\mathbf{R}}$ and the determinant of the Hermitian sample covariance matrix $\widehat{\boldsymbol{\Gamma}}$. In the next section, we will endow this observation with some intuitively appealing interpretation.

3. INTERPRETATION

How compelling is the GLRT decision rule derived above? Propriety is preserved under linear transformation, which includes rotation and scaling. Hence, the decision rule should be *invariant* to linear transformations applied to the sample matrix $\mathbf{S}$. That is, for nonsingular $\mathbf{T} \in \mathbb{C}^{N\times N}$ and

$$\widetilde{\mathbf{S}} = \mathbf{T}\mathbf{S} \Longleftrightarrow \widetilde{\boldsymbol{\Sigma}} = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^* \end{bmatrix}\boldsymbol{\Sigma},$$

the GLRs for $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ should be identical. This means that the GLR must be a function of a complete, or maximal, set of invariants for the sample augmented covariance matrix $\widehat{\mathbf{R}}$ under the transformation group

$$\mathcal{T} = \left\{\mathbf{T}_L = \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^* \end{bmatrix}, \det \mathbf{T} \neq \mathbf{0}\right\}$$

with group action

$$\widehat{\mathbf{R}} \longrightarrow \mathbf{T}_L \widehat{\mathbf{R}} \mathbf{T}_L^H.$$

Such a complete set of invariants is given by the *canonical correlations* [8, 9] computed from $\widehat{\mathbf{R}}$. We emphasize, however, the important constraint that in $\mathbf{T}_L$, the southeast block, $\mathbf{T}^*$, must be the conjugate of the northwest block, $\mathbf{T}$.

We will first assume that we know the augmented covariance matrix $\mathbf{R}$. Let $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{1/2}\boldsymbol{\Gamma}^{H/2}$ be the Cholesky factorization of $\boldsymbol{\Gamma}$, which means that $\boldsymbol{\Gamma}^{1/2}$ is generally not symmetric.

In order to find the canonical correlations, we determine the singular value decomposition (SVD) of the *coherence matrix* [9]

$$\mathbf{M} = \mathbf{\Gamma}^{-1/2}\mathbf{C}\mathbf{\Gamma}^{-T/2}$$

as

$$\mathbf{M} = \mathbf{F}\mathbf{K}\mathbf{G}^H. \tag{5}$$

Since $\mathbf{M}$ is complex symmetric, $\mathbf{M} = \mathbf{M}^T$, but not Hermitian symmetric, $\mathbf{M} \neq \mathbf{M}^H$, we have $\mathbf{F} = \mathbf{G}^*$, and the SVD (5) becomes Takagi's factorization [10],

$$\mathbf{M} = \mathbf{F}\mathbf{K}\mathbf{F}^T.$$

The matrix $\mathbf{F}$ is unitary and $\mathbf{K} = \operatorname{diag}(k_1, k_2, ..., k_N)$ contains the *canonical correlations* k_n on its diagonal. These satisfy $0 \leq k_n \leq 1$. The squared canonical correlations k_n^2 are the eigenvalues of the squared coherence matrix

$$\mathbf{M}\mathbf{M}^H = \mathbf{\Gamma}^{-1/2}\mathbf{C}\mathbf{\Gamma}^{-*}\mathbf{C}^*\mathbf{\Gamma}^{-H/2},$$

or equivalently, of the matrix $\mathbf{\Gamma}^{-1}\mathbf{C}\mathbf{\Gamma}^{-*}\mathbf{C}^*$ because

$$\mathbf{K}\mathbf{K}^H = \mathbf{F}^H\mathbf{\Gamma}^{-1/2}\mathbf{C}\mathbf{\Gamma}^{-*}\mathbf{C}^*\mathbf{\Gamma}^{-H/2}\mathbf{F}.$$

These eigenvalues are invariant to the choice of a square root for $\mathbf{\Gamma}$. It is also easy to show that the canonical correlations $\{k_n\}$ are invariant under the transformation group $\mathcal{T}$.

In order to compute the GLR (2), we use an estimate of the canonical correlation matrix, $\widehat{\mathbf{K}}$, from the sample Hermitian covariance matrix $\widehat{\mathbf{\Gamma}}$ and sample complementary covariance matrix $\widehat{\mathbf{C}}$. We then have

$$\begin{aligned}\ell &= \det(\mathbf{I} - \widehat{\mathbf{K}}\widehat{\mathbf{K}}^H) \\ &= \prod_{n=1}^{N}(1 - \hat{k}_n^2).\end{aligned} \tag{6}$$

Thus, the GLR is indeed a function of the estimated (squared) canonical correlations only. Even though we would use (3) and not actually compute canonical correlations in a practical application, this result provides an interesting interpretation.

The canonical correlations $\{k_n\}$ measure the correlations between the white, unit-norm canonical coordinates

$$\widetilde{\mathbf{s}} = \mathbf{F}^H\mathbf{\Gamma}^{-1/2}\mathbf{s}$$

and their conjugates

$$\widetilde{\mathbf{s}}^* = \mathbf{F}^T\mathbf{\Gamma}^{-*/2}\mathbf{s}^*.$$

More precisely, the canonical correlations $\{k_n\}$ are the cosines of the canonical angles between the linear subspaces spanned by $\widetilde{\mathbf{s}}$ and the complex conjugate $\widetilde{\mathbf{s}}^*$ [9]. If these angles are small, then $\mathbf{s}^*$ may be *linearly* estimated from $\mathbf{s}$, indicating that $\mathbf{s}$ is improper (obviously, $\mathbf{s}^*$ can be perfectly estimated from $\mathbf{s}$ if *widely linear* operations are allowed). If these angles are large, then $\mathbf{s}^*$ may not be *linearly* estimated from $\mathbf{s}$, indicating a proper $\mathbf{s}$. These angles are invariant to nonsingular linear transformations $\mathbf{T}$, thus validating our intuition that the internal coordinate system does not matter.

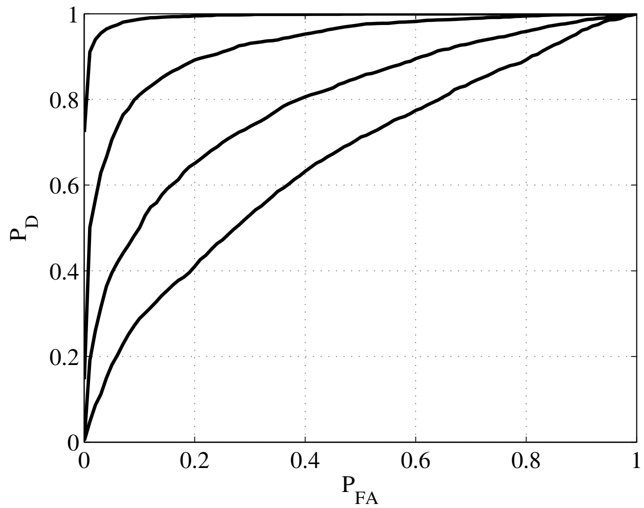

Fig. 1. Receiver operator characteristic (ROC) curves for the GLRT detector in the example. From northwest to southeast, these curves correspond to a phase tracking error variance of 0.7, 0.9, 1.1, and 1.3. The SNR for all curves is 0 dB.

4. AN EXAMPLE

In the scalar case $N = 1$, the GLRT decision rule becomes especially simple,

$$\ell = 1 - \frac{|\hat{c}|^2}{\hat{\gamma}^2}, \tag{7}$$

with sample covariance $\hat{\gamma}$ and sample complementary covariance $\hat{c}$. As a simple example, we consider the transmission of equiprobable binary data $b_m \in \{\pm 1\}$ over an AWGN channel that also rotates the transmitted bits by ϕ_m. The received statistic is

$$r_m = b_m e^{j\phi_m} + n_m,$$

where n_m are samples of white Gaussian noise and ϕ_m are samples of the channel phase. We are interested in classifying this channel as either noncoherent or (at least partially) coherent. That is: Is this channel rotationally invariant or can some phase information be extracted from the received statistic r_m? Even though r_m is not Gaussian, the GLR (7) is well suited for this hypothesis test.

We evaluate the performance of the GLRT detector by Monte Carlo simulations. Under hypothesis 0, we assume that the phase samples ϕ_m are i.i.d. and uniformly distributed, and under hypothesis 1, we assume that the phase samples are i.i.d. and drawn from a Gaussian distribution. This means that under H_0, no useful phase information can be extracted, whereas under H_1, a phase estimate is available, albeit with a tracking error.

Figure 1 plots the probability of detection, P_D, vs. the probability of a false alarm, P_{FA}, for a signal-to-noise ratio of

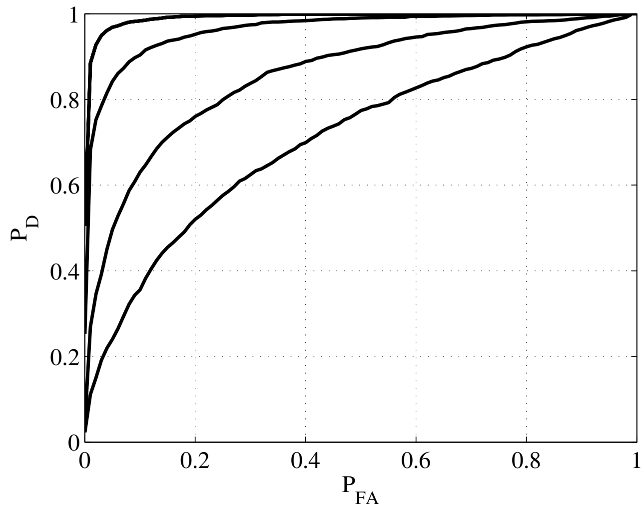

Fig. 2. ROC curves for the GLRT detector in the example. From northwest to southeast, these curves correspond to a SNR of 6, 3, 0, and -3 dB. The phase tracking error variance for all curves is 1.

0 dB and varying variance for the phase tracking error. Figure 2 plots this curve for varying signal-to-ratios and a phase tracking error with variance 1. In all cases, the number of samples was taken as $M = 1000$.

Not surprisingly, we can see from these figures that the test performs very well for larger SNR or for smaller phase tracking error.

5. CONCLUSIONS

We have presented a GLR hypothesis test for impropriety. The reason this GLRT is compelling is that it has the right invariance. It is invariant under linear, but not widely linear, transformation just as propriety is preserved under linear transformation, but not widely linear, transformation. In fact, we have shown that the GLR is a function of the squared canonical correlations between the data and its complex conjugate only. These canonical correlations are a set of maximal invariants for this problem. It remains to be shown whether the proposed GLRT is also a uniformly most powerful test.

6. REFERENCES

[1] A. van den Bos, "The multivariate complex normal distribution—a generalization," *IEEE Trans. Inform. Theory*, vol. 41, no. 2, pp. 537 – 539, Mar. 1995

[2] B. Picinbono, "Second-order complex random vectors and normal distributions," *IEEE Trans. Signal Processing*, vol. 44, no. 10, pp. 2637 – 2640, Oct. 1996

[3] G. Gelli, L. Paura, A. R. P. Ragozini, "Blind widely linear multiuser detection," *IEEE Commun. Lett.*, vol. 4, pp. 187 – 189, 2000.

[4] A. Lampe, R. Schober, W. Gerstacker, "A novel iterative multiuser detector for complex modulation schemes," *IEEE J. Sel. Areas Commun.*, vol. 20, pp. 339 – 350, 2002.

[5] P. J. Schreier, L. L. Scharf, "Second-order analysis of improper complex random vectors and processes," *IEEE Trans. Signal Processing*, vol. 51, no. 3, pp. 714 – 725, Mar. 2003

[6] B. Picinbono, P. Chevalier, "Widely linear estimation with complex data," *IEEE Trans. Signal Processing*, vol. 43, no. 8, pp. 2030 – 2033, Aug. 1995

[7] K. V. Mardia, J. T. Kent and J. M. Bibby, *Multivariate Analysis*, New York, NY: Academic Press, 1979

[8] H. Hotelling, "Analysis of a complex pair of statistical variables into principal components," *J. Educ. Psychol.*, vol. 24, pp. 417 – 441, 498 – 520, 1933

[9] L. L. Scharf, C. T. Mullis, "Canonical coordinates and the geometry of inference, rate, and capacity," *IEEE Trans. Signal Processing*, vol. 48, no. 3, pp. 824 – 831, Mar. 2000

[10] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge, UK: Cambridge University Press, 1985