ON THE ROOTS OF THE 3X3 MEDIAN FILTER

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ABSTRACT

A characterization of the roots of the 3x3 median filter is given. We define the properties of *local smoothness* and of *local roughness* for the roots of the 3x3 median filter. Roots that are locally rough everywhere are binary and periodic; otherwise, unlike the 1D case, a root may be non binary or non periodic. This partially generalizes to dimension 2 the results of Brandt [1] and Tyan [2]; in particular, everywhere local smoothness may be interpreted as *local monotonicity* in dimension 2. We concentrate on the binary roots of the filter with the 3x3 window shape; the complexities of the general problem of characterizing the roots of the 2D median filter makes this an acceptable starting point.

1. INTRODUCTION

An *image,* i.e. a 2D-signal, is a function s : **Z**x**Z** → **R**, where **Z** is the set of the integers and **R** is the set of the real numbers. Each d∈D is a *pixel* and s(d) is the *value* or the *color* of the image at the pixel d.

The roots of the 1D median filter have been characterized and classified, particularly by Tyan [2] and Brandt [1]. With reference to a window of length w = 2k+1, a 1D signal is said to be *smooth* if it is *locally monotonic* of degree k+2 [2]. Smooth signals are roots of the corresponding median filter. A signal is *rough* if it is not smooth [3]. Rough roots are binary and periodic [1]; in particular, if a root has a non-monotonic segment of length k+2 then it has no monotonic segment of such a length. Astola, Heinonen and Neuvo [4] have shown cases of binary, doubly periodic signals for a generic filter of window size NxN, based on periodic 1D rough roots.

With the smoothness criterion of local monotonicity, an optimal smoother of 1D signals is given by *locally monotonic regression* [5], [6], [7]. A characterization of the 2D roots into smooth and rough roots provides a criterion and smoothness for images; along these lines, 2D binary locally monotonic regression was proposed in [8].

In this paper, we concentrate on the binary roots of the filter with the 3x3 window shape. The complexities of the general problem of characterizing the roots of the 2D median filter makes this an acceptable starting point.

Given an image, a *sister* of a pixel is a neighbor pixel of the same color, using 8-pt. connectivity; a set of four sisters of a pixel is said to be a *family* of the set. A *block* is a largest set of connected pixels having the same color. In this way each image determines a partition of its domain into largest connected subsets of pixels.

2. 2D BINARY ROOTS

In the 2D case, when the window is shifted by just one pixel, several pixels enter the window (and the same amount leave the window) [2]. Also, there are 8 possible directions along which the window can be moved.

A binary signal is a root if and only if each pixel has at least 4 connected pixels; correspondingly, we define an *atomic tile* as a segment of size 3x3 with an invariant central pixel. The *weight* of an atomic tile is the number of pixels with value equal to that of the center; thus, for binary roots, the weight must be at least 5.

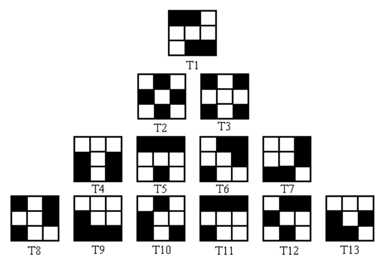


Fig.1. Molecular binary atomic tiles T1-T13, of minimal weight (k+1 = 5 , modulo inversion, rotations and reflection.

Without regard to inversion or negation (black↔white), rotations or reflection, there are 13 tiles of weight 5, as shown in *Fig. 1*. For weights greater than 5, for a central white pixel, we get the tiles in *Fig. 2*, modulo rotations, reflections and inversions.

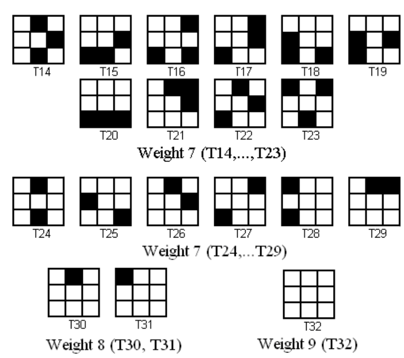


Fig. 2. Molecular binary tiles T14-T32, of non-minimal weight

3. FORBIDDEN SEGMENTS

Some segments are part of no root. This may be seen as an implication of the following lemmas.

Lemma 1. The segment indicated in *Fig. 3*, or any of its (reflected, negated or rotated) versions, cannot be part of a root.
Proof. The black pixel of the segment cannot have 4 sisters.
Corollary. None of Tiles T5, T19, T23, T24, T25, T30 can be part of a root.

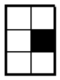
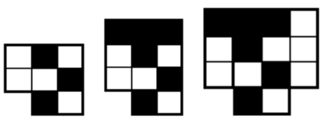


Fig. 3. The segment shown cannot be part of a root.

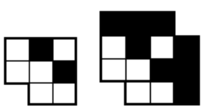
Lemma 2. The segment indicated in *Fig. 4(a)*, or any of its versions, cannot be part of a root.
Proof. The black pixel above in the tile forces the situation in *Fig.4(b),* where the white pixel on the right and above forces the situation on *Fig. 4(c);* but now the black pixel on the right below cannot have 4 sisters.
Corollary. None of Tiles T12 or T14, or any of their versions, can be part of a root.



(a) (b) (c)

Fig. 4. The segment shown in (a) cannot be part of a root.

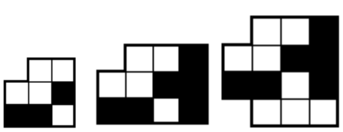
Lemma 3. The segment indicated in *Fig. 5(a),* or any of its versions, cannot be part of a root.
Proof. The black pixels force the situation in *Fig. 5(b),* where the white pixel on the right and above cannot have four sisters.
Corollary. None of Tiles T22, T26, nor any of their versions, can be part of a root.



(a) (b)

Fig. 5. The segment shown in (a) cannot be part of a root.

Lemma 4. The segment indicated in *Fig. 6(a),* or any of its (reflected, negated or rotated) versions, cannot be part of a root.
Proof. The black pixel on the right in the segment forces the situation in *Fig. 6(b)* where the white pixel below forces the situation in *Fig. 6(c),* now the black pixel below and second from right cannot have 4 sisters.
Corollary. None of Tiles T10, or T15, or any of their versions, can be part of a root.



(a) (b) (c)

Fig. 6. The segment shown in (a) cannot be part of a root.

4. EXPANSIONS OF TILES

Given an atomic tile, or more generally, given a segment, in which roots may it appear? We call such roots the *expansions* of the tile. The expansions are characterized by the lemmas and examples in this section.

Lemma 5. There is only one root that contains the segment shown in *Fig. 7.* Besides, the root is periodic.
Proof. The white pixel on the right in the segment requires at least two sisters on a new column at the right. Three white pixels are not possible because of *Lemma 1*; also because of *Lemma 2*, two sisters are possible only if separated by a white pixel, as in *Fig. 8(a).* Repeating the argument on the white pixels of the resulting segment *Fig. 8(a),* the segment in *Fig. 8(b)* results; the new expanded segment must be completed as in *Fig. 8(c).*
Corollary. The only root that contains Tile T2 as a segment is the periodic signal illustrated in *Fig. 9(a),* with period illustrated in *Fig. 9(b).*

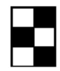
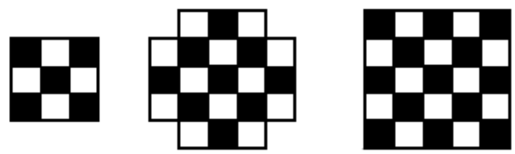
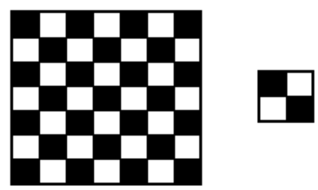


Fig. 7. The segment shown only can be part of one root.



(a) (b) (c)

Fig. 8. The only expansion of Tile T2.



(a) (b)

Fig. 9. Only the root in (a) contains Tile T2. (b) a period of the root.

Some examples of expansions of Tiles T1, T2, T3, T4, T6, T7, T8, T9, T11, T13, T16, T17, T18, T20, T21, T27, T28, T29, T31 and T32 are shown in *Fig.10-22* .

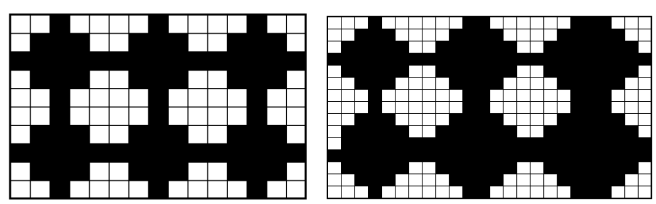


Fig. 10. Examples of roots containing tiles T4, T9, T21, T31, T32.

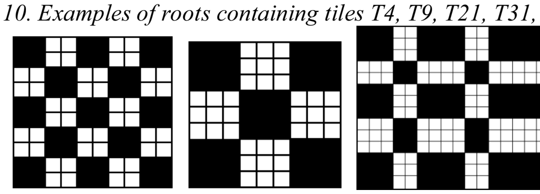


Fig. 11. Examples of roots containing tiles T7, T20, T32.

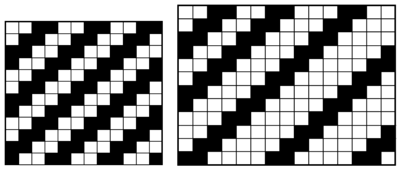


Fig. 12. Examples of roots containing tile T6.

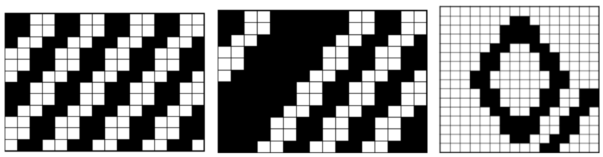


Fig. 13. Examples of roots containing tile T17.

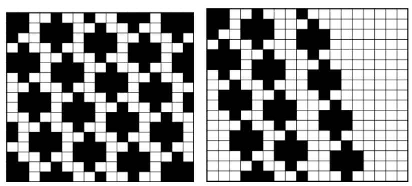


Fig. 14. Examples of roots containing tile T8.

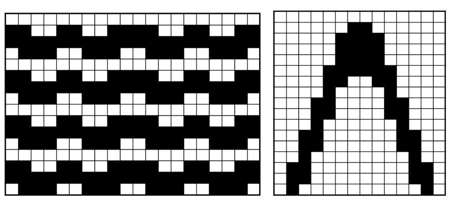


Fig. 15. Examples of roots containing tile T11.

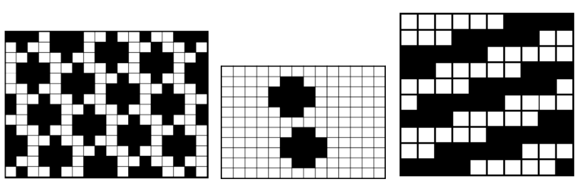


Fig. 16. Examples of roots containing tile T1.

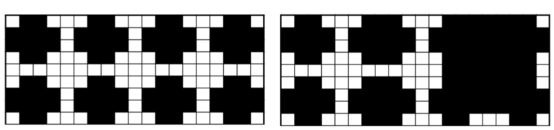


Fig. 17. Examples of roots containing tile T28.

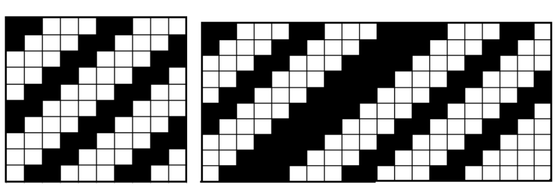


Fig. 18. Examples of roots containing tile T27.

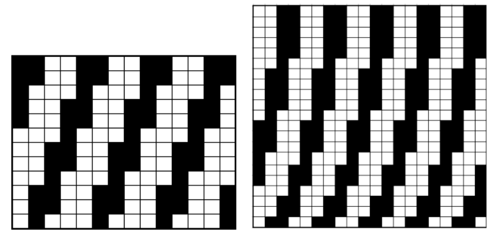


Fig. 19. Examples of roots containing tile T29.

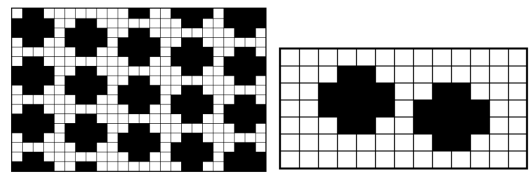


Fig. 20. Examples of roots containing tile T18.

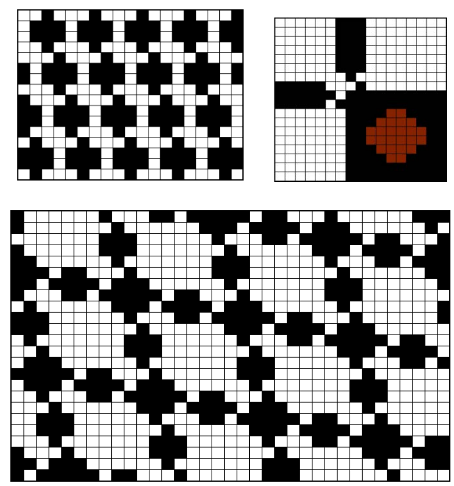


Fig. 21. Examples of roots containing tiles T13, T16.

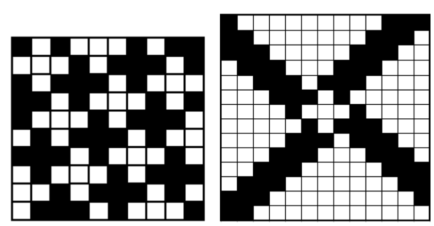


Fig. 22. Examples of roots containing tile T3.

Lemma 6. The only atomic tiles that may appear on a root are Tiles T1, T2, T3, T4, T6, T7, T8, T9, T11, T13, T16, T17, T18, T20, T21, T27, T28, T29 T31 and T32.

Proof. By the previous examples *Fig.10-22* and by *Lemmas 1-4*.

Lemma 7. The only root that contains a version of Tile T2 is periodic. The remaining tiles in *Lemma 6* may appear both in periodic and in non periodic roots.

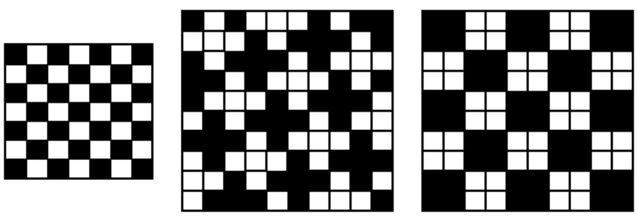
Proof. By the previous examples *Fig.10-22* and by the corollary to *Lemma 5*.

5. LOCAL ROUGHNESS/SMOOTHNESS

A root is said to be *locally rough* at pixel (i, j) if the atomic tile centered at such a pixel turns out to be a version of tiles T2, T3, T7 or T13. Otherwise (if the tile in question is a version of tiles T1, T4, T6, T8, T9, T11; T16, T17, T18, T20, T21; T27, T28, T29; T31 or T32), the signal is said to be locally smooth at the pixel. Clearly, a root is locally smooth at a pixel if and only if it is not locally rough there. A root is *smooth* if is every where smooth, i.e. if it does not contain a version of the following segment ("an x"):

Fig. 23. An x.

Lemma 8. If a root is everywhere rough then it is binary and periodic.
Proof. If T2 appears in a root, the root is binary and periodic by the corollary to *Lemma 5*. If an everywhere rough root contains Tile T7, it does not contain versions of T13 nor of T3 and in fact is binary and periodic as in *Fig. 24 (c)*.
There is no root containing only versions of Tile T3, nor only versions of Tile T13. The only root that contains versions of Tiles T3 and T13 is binary periodic, as in *Fig 24 (b)*. Thus the only possible roots that are everywhere rough are like those in *Fig. 24*, which are binary and periodic.



(a) *(b)* *(c)*

Fig .24. The only roots that are everywhere rought.

Lemma 9. If a root is everywhere rough, it consists of only two (infinite) blocks.
Proof. The x that is present at each of its tiles connects the black and white blocks.
Corollary. If a root has more than two bocks, it is not everywhere rough.

6. THE NON-BINARY CASE

If the partition into blocks of a binary root gives more that two blocks, it is possible to obtain a new root by redefining the image on one or more blocks.
From the examples, there are non-smooth roots that are not periodic, as in *Figs. 10, 11, 14, 21 and 22*. Also, not surprisingly, there are smooth roots that are not periodic, as in *Fig. 12, 13 and 15-20.*
The impeccable performance on images of the 2D median filter may be due in part to the existence of roots that are rough at some points and smooth at others.

7. CONCLUSION AND FURTHER RESEACH

We have taken steps towards a general classification of the root signals of the 2D median filter, a more complex problem than that in the 1D case, which was not trivial. This analysis of roots furthers

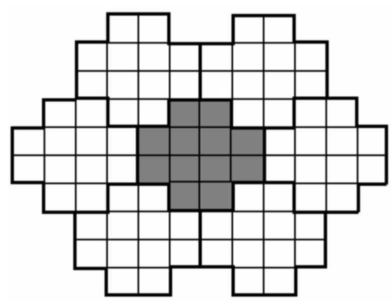


Fig 25. Partition into blocks of a smooth signal (each aztec pyramid may have a different color)

the understanding of the median filter; in spite of the wide use we make of the filter, it seems that more hard facts about it are yet to be found.
Unlike the 1D case where every root containing a non-monotonic segment of length k+2 contains no monotonic segment of that length, a 2D root may be both locally rough and locally smooth.

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