THRESHOLDING DISTANCE PLOTS USING TRUE RECURRENCE POINTS

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ABSTRACT

Recurrence plots (RP) visualize multi–dimensional state spaces and represent the recurrence of states of a system. Recurrence points can be divided into true recurrence points and false recurrence points (also called sojourn points). We introduce the true recurrence point recurrence plot, TRP, a variant of the traditional RP excluding the sojourn points. This is a cleaned up RP free from recurrence points originating from tangential motion, and hence a more robust representation of unstable periodic orbits. The method is demonstrated with three simple systems, a periodic sine wave, a quasi–periodic torus and the x–component of the chaotic Lorenz system.

1. INTRODUCTION

Recurrence plots (RP) were introduced to graphically display recurring patterns and non–stationarity in time series [1]. RPs can be used on rather short time series and represent the recurrence of states of a system (i.e. how often a small region in state space is visited). Unlike other methods such as Fourier, Wigner–Ville or wavelets, recurrence is a simple relation, which can be used for both linear and nonlinear data. Further it can be used for analysis of short data sets [2].

A distance plot is a symmetric matrix where each element *(i,j)* represents the distance between the states a_i and a_j in some state space. If the distance matrix is somehow thresholded to logical values, it is called a RP (hence a RP is a two–dimensional matrix with black and white dots, where black dots mark a recurrence) [3].

We present a new variation of RP based on the work of Gao et al [4]. By excluding sojourn points from the RP (also termed autocorrelation or tangential motion [3]), we get the True recurrence point Recurrence Plot, TRP. Our aim is to clean up RPs from recurrence points originating from tangential motion. We also hope that this tool will be better equipped to find unstable periodic orbits compared to ordinary RPs.

2. RECURRENCE PLOTS

The dynamics of a time discrete system is determined by its possible states in a multivariate vector space (called state space or phase space). The transitions between the states are described by vectors, and these vectors form a trajectory describing the time evolution of the system. An observed signal S is a projection from this multivariate state space onto a one–dimensional time series. S can be considered as a set of n scalar measurements, $S = \{s_1, s_2, \ldots, s_n\}$, from which a sequence of N d–dimensional vectors a_k can be constructed using Takens' delay embedding theorem, $a_k = \{s_k, s_{k+\tau}, s_{k+2\tau}, \ldots, s_{k+(d-1)\tau}\}$, $k = 1, 2, \ldots, N$, where τ is a delay parameter and d is the embedding dimension [5]. The purpose of the embedding is to unfold the projection back into a reconstructed state space that is dynamically and topologically equivalent to the state space that generated the time series S [6]. Since the dynamics of the reconstructed state space contains the same topological information as the original state space, characterization or prediction based on the reconstructed state space are as valid as if they were made in the true state space. In this study, τ and d were chosen based on the standard techniques of average mutual information and false nearest neighbors [5].

The dimension of a state space is usually higher than two or three, making it hard to visualize. However, RPs enable us to analyse d–dimensional state space trajectories through a two–dimensional representation of its recurrences. An RP is usually defined as

$$R_{i,j} = \Theta\left(r_i - \left\| a_i - a_j \right\|\right) \quad i, j = 1,2,\ldots,N \tag{1}$$

where r_i is a threshold distance, $\|\bullet\|$ is a norm and $\Theta(\bullet)$ is the Heaviside function. Any norm could be used, but usually the L_1–norm, the L_2–norm or the L_∞–norm is used [3].

The most commonly used neighborhood is defined with a fixed radius $r_i = r, \quad \forall i$. However, in the original definition of RPs, r_i was allowed to vary in such a way that a fixed amount of neighbors were chosen. In such a setting, all columns in the RP will have the same recurrence density. A number of further variations of RPs can be found in the literature. The definition of closeness can be modified by requiring $r_i \leq \|a_i - a_j\| \leq r_j$. An advantage of such a corridor thresholded recurrence plot is its increased robustness against recurrence points coming from tangential motion. However, the threshold corridor removes the inner points in broad diagonal lines, resulting in two lines instead of one [7]. Another approach that also attempts to clean up RPs from recurrence points originating form tangential motion is the perpendicular RP [8]. This recurrence plot contains only those points a_j that fall into the neighborhood of a_i and lie in the (d−1)–dimensional subspace that is perpendicular to the trajectory at a_i. These points correspond locally to those lying on a Poincaré section [3]. There is no best choice when choosing method to calculate the RP, and the selection should be based on the problem at hand and on the kind of data available.

3. THRESHOLDING BASED ON TRUE RECURRENCE POINTS

Our approach is similar to the two last described. We also try to avoid contributions from false recurrence points by excluding the sojourn points described by Gao et al [4]. The recurrence points enclosed by the hypersphere in (1) can be divided in two classes; true recurrence points and sojourn points, see Fig. 1.

For each state a_i on the reconstructed trajectory, a neighborhood of radius r can be defined as a set of points:

$$B_r(a_i) = \{a : \|a - a_i\| \leq r\} \tag{2}$$

The true recurrence points are defined as the set of first trajectory points entering the neighborhood from the outside. These are denoted with filled circles in Fig. 1. The trajectory may stay inside the neighborhood for a while, thus generating a sequence of points, seen as open circles in Fig. 1. These are called sojourn points [4]. It is clear that there will be more such points when the size of the neighborhood gets larger as well as when the trajectory is sampled more densely. Sojourn points form vertical and horizontal lines, thus square textures, in recurrence plots defined by (1).

Denote the subset of recurrence points in $B_r(a_i)$ by $S_1=\{a_{t1}, a_{t2}, \ldots, a_{ti}, \ldots\}$ and the corresponding Poincaré recurrence times as $\{T(i)=t_{i+1}-t_i,\ i=1,2,\ldots\}$. Note the time is computed based on successive returns, not on the returning points and the reference point. This way, the nearest neighbors of the reference point a_i are assigned a natural time ordering, and the reference point does not play a central role. We also note $T(i)$ may be 1 for some i. This occurs when there is at least one sojourn point. Existence of such points makes further quantitative analysis difficult, since the number of sojourn points depends on the embedding parameters, the sampling rate and the neighborhood size. Thus, we remove the sojourn points from the set S_1 (which can easily be achieved by monitoring whether the recurrence times of the first type are 1 or not), and let the remaining set be denoted by $S_2=\{a'_{t1}, a'_{t2}, \ldots, a'_{ti}, \ldots\}$.

Both S_1 and S_2 are dependent on the reference point a_i, and we denote this dependence as $S_1(a_i)$ and $S_2(a_i)$, respectively. To create a RP free from sojourn points, we can calculate the true recurrence points of each state a_i along the trajectory, and mark their occurrence in an NxN matrix:

$$R_{i,j} = \begin{cases} 1 & a_j \in S_2(a_i) \\ 0 & else \end{cases} \tag{3}$$

Basically, this is implemented as a double loop. The outer loop goes through each state a_i along the trajectory and calculates the set $S_2(a_i)$. The inner loop sets the value of row (j) in column (i) to 1 if a_j is a true recurrence point to a_i (i.e. is in the set $S_2(a_i)$).

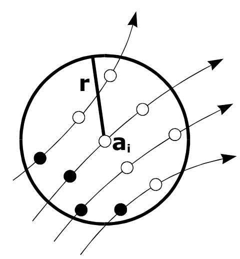

Fig. 1. The two kinds of recurrence points, true recurrence points (filled circles) and sojourn points (open circles).

4. EVALUATION

The TRP is compared with the traditional RP in (1) using three different systems; a sine wave (4), a quasi–periodic torus (5) and the x–component of the chaotic Lorenz system (6). The sampling time was 0.01 in (4) − (5) and 0.1 in (6), and all time series were normalized to unity. The parameters used were $\tau = 1$, $d = 3$ and $r = 0.1$. The resulting RPs are illustrated in Fig. 2.

$$x(t) = \sin(2\pi t) \tag{4}$$

$$x(t) = \sin(2\pi t) + \sin(2\sqrt{2}\pi t) \tag{5}$$

$$\begin{aligned} \partial x/\partial t &= 0.1(x-y) \\ \partial y/\partial t &= -0.1x - y - xz \\ \partial z/\partial t &= xy - 0.02z \end{aligned} \tag{6}$$

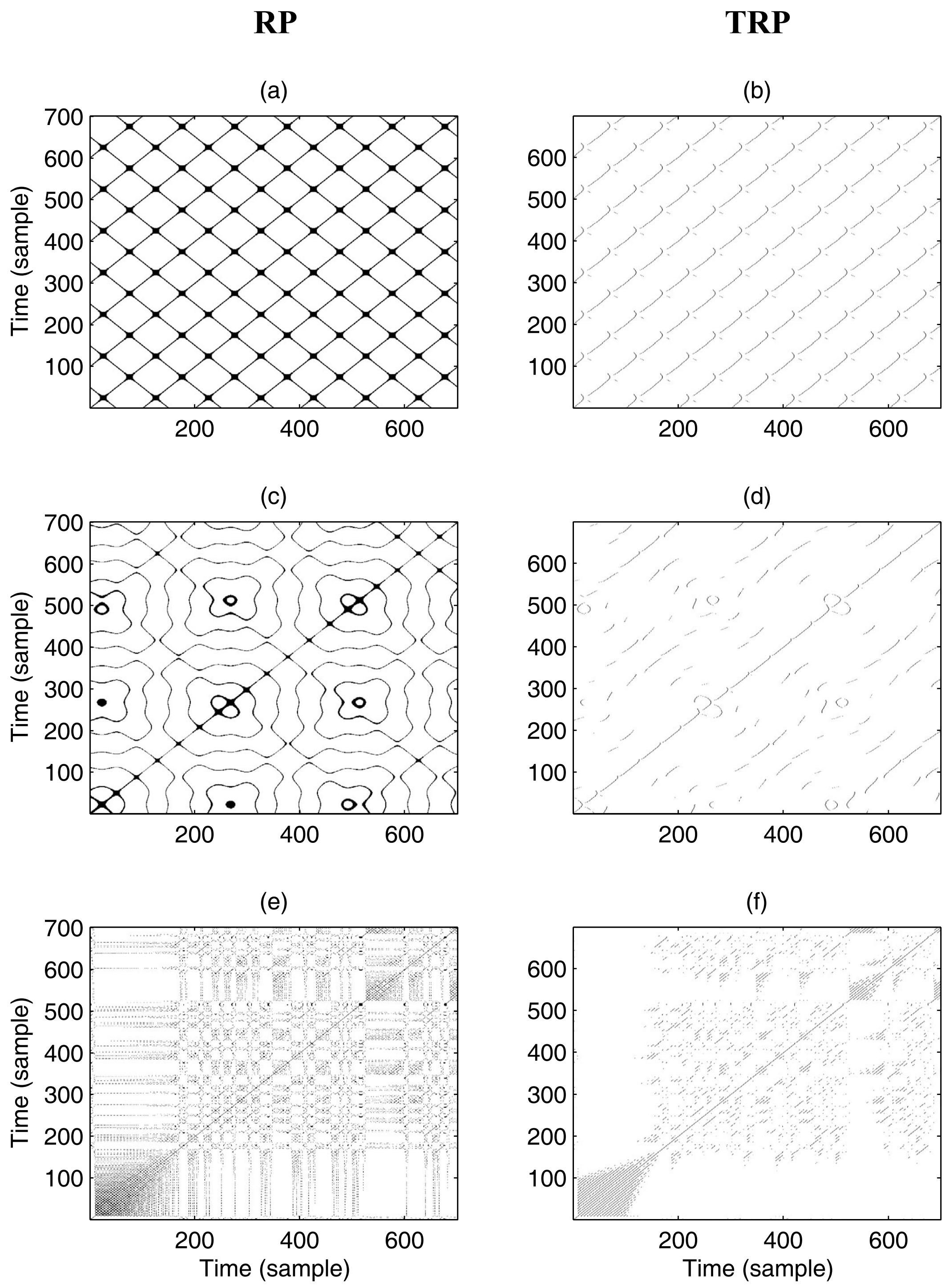

Fig. 2. Recurrence plots generated from a sine wave (a,b), a torus (c,d) and a Lorenz system (e,f). The left column was created with the definition in (1) while the right column was calculated with our approach, avoiding the sojourn points (vertical and horizontal line segments).

As mentioned in the previous section, and as can be seen in Fig. 2, sojourn points form vertical and horizontal lines in traditional RPs. This gives the structures in the plot a certain width and height. The TRP on the other hand is composed of thin lines that don't stretch out in various directions.

5. DISCUSSION

There are several methods trying to characterize chaotic systems. One of the most used is the largest Lyapunov exponent, LLE, which is a measure of the separation rate of close trajectories in state space. There are different methods to estimate the LLE, but they often require large stationary data sets, something that is not always available (for instance in biomedical signals). RPs are also used to investigate chaotic systems, and they can be used on rather short time series. When pairs of points on different orbits keep close together for some time interval, they form diagonal line segments in the RP. For chaotic systems, such closeness breaks down exponentially fast. This indicates that the lengths of diagonal line segments (due to this mechanism) are related to the largest positive Lyapunov exponent [4]. Since diagonal lines have a certain width and height given by the sojourn points, ordinary RPs are not suited for calculations of LLE. Even when the cutoff distance is taken small enough, the RP pattern spreads out in various directions, and if the threshold is set too small, the diagonal structures could break into smaller subsegments. In contrast, the TRP is more stable since the spurious widths are implicitly removed by the method.

Problems with removing the sojourn points could arise as the neighborhood is defined with a finite threshold r. This is apparent when the traces contain small cycles which look like tangential motion. It is also important to note that some recurrence quantification measures become pointless when using the TRP. For example, both laminarity and trapping time are based on vertical (and horizontal) structures which we remove.

6. CONCLUSION

We have introduced the True recurrence point Recurrence Plot, TRP. The TRP is a variant of the traditional RP with the difference that sojourn points are left out of the plot. Particularly, the method cleans up RPs, making them free from recurrence points originating from tangential motion.

ACKNOWLEDGMENT

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7. REFERENCES

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