EFFECTS OF COORDINATE SHIFTS ON TLS ESTIMATION BIAS IN BEARINGS-ONLY TARGET LOCALIZATION PROBLEMS

Kutluyıl Doğançay

University of South Australia
School of Electrical and Information Engineering
Mawson Lakes, SA 5095, Australia

ABSTRACT

This paper analyzes the effects of local coordinate shifts on the resulting estimation bias for the total least squares (TLS) bearings-only target location algorithm. The TLS estimator has been proposed to alleviate the severe bias problems associated with the traditional pseudolinear estimator. An interesting property of the TLS estimator is that its bias is affected by changes in the origin of the local Cartesian coordinates. This is formally proven and demonstrated in the paper. Suggestions are also provided as to how to reduce the TLS estimation bias through local coordinate shifts.

1. INTRODUCTION

The objective of bearings-only target localization is to estimate the location of a target by utilizing a sequence of bearing measurements collected by a moving observer or fixed observers at distinct locations. The pseudolinear estimator [1] provides a simple closed-form solution to the bearings-only target localization problem. Despite its low complexity and the absence of convergence problems, the pseudolinear estimator suffers from large estimation bias due to the correlation between the measurement matrix and the bearing noise. The bias of the pseudolinear estimator has been studied in the target tracking and localization literature (see e.g. [1, 2]). To overcome this bias problem, various fixes have been proposed based on batch iterative and closed-form instrumental variables [1, 3, 4], and total least squares (TLS) [5, 6]. Unlike the pseudolinear estimator, the TLS estimator attempts to correct the errors in both the measurement matrix and the data vector. This generally results in improved bias performance.

In this paper we provide formal proofs for the dependence of the TLS estimation bias on the target localization geometry, in particular local coordinate shifts for a given geometry. These proofs are based on preliminary observations made in [6] regarding bias variations with geometry translations involving rotations and/or shifts. Some suggestions are also provided as to how the origin of local coordinates should be selected in order to reduce the TLS estimation bias. The formal results are backed up with simulation examples.

2. PASSIVE BEARINGS-ONLY TARGET LOCALIZATION

The two-dimensional passive target localization problem using bearing measurements is depicted in Fig. 1 where $\boldsymbol{p}$ is the location of a stationary target, and θ_k and $\boldsymbol{r}_k$ are the bearing angle and observer position, respectively, at time instant k. The relationship between θ_k, $\boldsymbol{r}_k$ and $\boldsymbol{p}$ is given by the nonlinear equation:

$$\theta_k = \tan^{-1}\frac{\Delta y_k}{\Delta x_k}, \quad k = 1, \ldots, N \tag{1}$$

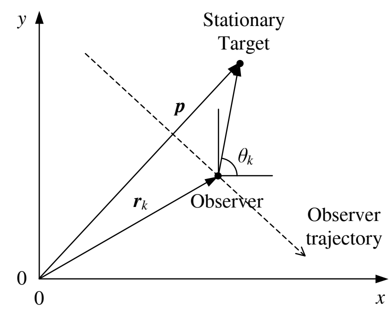

Fig. 1. Two-dimensional bearings-only target localization geometry.

where $\Delta y_k = p_y - r_{y,k}$, $\Delta x_k = p_x - r_{x,k}$, $\boldsymbol{p} = [p_x, p_y]^T$ and $\boldsymbol{r}_k = [r_{x,k}, r_{y,k}]^T$. Here ^{T} denotes matrix transpose.

The objective of target localization is to estimate the target location $\boldsymbol{p}$ from a sequence of bearing measurements over the interval $1 \le k \le N$. In practice, the bearing measurements are modelled as

$$\tilde{\theta}_k = \theta_k + n_k, \quad k = 1, \ldots, N \tag{2}$$

where the $\tilde{\theta}_k$ are the bearing measurements and n_k is zero-mean white Gaussian bearing noise with variance $\sigma^2_{n_k}$. We assume that the target is observable from the available observer positions and bearing measurements.

The nonlinear relationship between the bearing angles and the receiver locations can be formulated as a linear matrix equation [2]:

$$\boldsymbol{A}\boldsymbol{p} = \boldsymbol{b} + \boldsymbol{\eta} \tag{3}$$

where

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \vdots \\ \boldsymbol{a}_N^T \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} \boldsymbol{a}_1^T\boldsymbol{r}_1 \\ \boldsymbol{a}_2^T\boldsymbol{r}_2 \\ \vdots \\ \boldsymbol{a}_N^T\boldsymbol{r}_N \end{bmatrix}, \quad \boldsymbol{\eta} = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_N \end{bmatrix}. \tag{4}$$

Here $\boldsymbol{a}_k = [\sin\tilde{\theta}_k, -\cos\tilde{\theta}_k]^T$ and $\eta_k = \|\boldsymbol{p} - \boldsymbol{r}_k\|_2 \sin n_k$. The least squares solution to $\boldsymbol{A}\boldsymbol{p} \approx \boldsymbol{b}$, given by $\hat{\boldsymbol{p}}_{\text{PLE}} = (\boldsymbol{A}^T\boldsymbol{A})^{-1}\boldsymbol{A}^T\boldsymbol{b}$, is referred to as the pseudolinear estimator (PLE).

3. TLS LOCATION ESTIMATOR

The PLE estimator exhibits large bias. It may be possible to reduce the estimation bias significantly by employing TLS to solve $\boldsymbol{A}\boldsymbol{p} \approx \boldsymbol{b}$ in (3) [5]. Central to TLS is the concept of perturbing both $\boldsymbol{A}$ and $\boldsymbol{b}$

in a minimal fashion, rather than $\boldsymbol{b}$ only as in the case of LS estimation, to obtain a consistent matrix equation that relates the perturbed $\boldsymbol{A}$ to the perturbed $\boldsymbol{b}$. TLS aims to solve the following constrained minimization problem [7, 8]

$$[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}] = \underset{\boldsymbol{b}+\boldsymbol{\delta} \in \text{Range}(\boldsymbol{A}+\boldsymbol{\Delta})}{\arg\min} \|\boldsymbol{L}[\boldsymbol{\Delta}, \boldsymbol{\delta}]\boldsymbol{T}\|_F \tag{5}$$

where $\boldsymbol{L}$ and $\boldsymbol{T}$ are nonsingular diagonal weighting matrices

$$\boldsymbol{L} = \text{diag}(l_1, l_2, \ldots, l_N)$$
$$\boldsymbol{T} = \text{diag}(t_1, t_2, t_3)$$

and $\|\cdot\|_F$ denotes the Frobenius norm defined by

$$\|\boldsymbol{H}\|_F = \left(\sum_i \sum_j |h_{ij}|^2\right)^{1/2}.$$

The TLS solution is given by $\hat{\boldsymbol{p}}_{\text{TLS}}$ which satisfies

$$(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})\hat{\boldsymbol{p}}_{\text{TLS}} = \boldsymbol{b} + \hat{\boldsymbol{\delta}} \tag{6}$$

where $\hat{\boldsymbol{\Delta}}$ and $\hat{\boldsymbol{\delta}}$ are the minimal TLS perturbations defined in (5) [9].

Next we consider how to obtain $\hat{\boldsymbol{\Delta}}$, $\hat{\boldsymbol{\delta}}$ and $\hat{\boldsymbol{p}}_{\text{TLS}}$. The TLS estimate defined in (6) can be obtained from the singular value decomposition (SVD) of the weighted augmented matrix [9]

$$\boldsymbol{L}[\boldsymbol{A}, \boldsymbol{b}]\boldsymbol{T} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T = \sum_{i=1}^{3} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \tag{7}$$

where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$ are the singular values, and $\boldsymbol{U} = [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3]$ and $\boldsymbol{V} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3]$ are orthogonal matrices, i.e., $\boldsymbol{U}^T\boldsymbol{U} = \boldsymbol{I}$ and $\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$. We shall assume $\sigma_2 > \sigma_3$, which avoids the complications involved with finding a TLS solution for repeated singular values. The perturbations $\hat{\boldsymbol{\Delta}}$ and $\hat{\boldsymbol{\delta}}$ minimizing (5) are obtained from a reduced rank approximation of $\boldsymbol{L}[\boldsymbol{A}, \boldsymbol{b}]\boldsymbol{T}$ [9]:

$$\boldsymbol{L}[\boldsymbol{A} + \hat{\boldsymbol{\Delta}}, \boldsymbol{b} + \hat{\boldsymbol{\delta}}]\boldsymbol{T} = \boldsymbol{L}[\boldsymbol{A}, \boldsymbol{b}]\boldsymbol{T} + \boldsymbol{L}[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}]\boldsymbol{T} = \sum_{i=1}^{2} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T \tag{8a}$$

$$[\boldsymbol{A} + \hat{\boldsymbol{\Delta}}, \boldsymbol{b} + \hat{\boldsymbol{\delta}}] = \boldsymbol{L}^{-1}\left(\sum_{i=1}^{2} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T\right)\boldsymbol{T}^{-1}. \tag{8b}$$

It follows from (7) and (8) that

$$\boldsymbol{L}[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}]\boldsymbol{T} = -\sigma_3 \boldsymbol{u}_3 \boldsymbol{v}_3^T \tag{9a}$$
$$[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}] = -\sigma_3 \boldsymbol{L}^{-1}\boldsymbol{u}_3 \boldsymbol{v}_3^T \boldsymbol{T}^{-1} \tag{9b}$$

where $\|\boldsymbol{L}[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}]\boldsymbol{T}\|_F = \sigma_3$. Using (8), (6) can be rewritten as

$$[\boldsymbol{A} + \hat{\boldsymbol{\Delta}}, \boldsymbol{b} + \hat{\boldsymbol{\delta}}]\begin{bmatrix}\hat{\boldsymbol{p}}_{\text{TLS}} \\ -1\end{bmatrix} = \boldsymbol{0} \tag{10a}$$

$$\boldsymbol{L}^{-1}\left(\sum_{i=1}^{2} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T\right)\boldsymbol{T}^{-1}\begin{bmatrix}\hat{\boldsymbol{p}}_{\text{TLS}} \\ -1\end{bmatrix} = \boldsymbol{0}. \tag{10b}$$

Noting that $\boldsymbol{v}_1^T\boldsymbol{v}_3 = \boldsymbol{v}_2^T\boldsymbol{v}_3 = 0$ because $\boldsymbol{V}$ is an orthogonal matrix, the solution of (10b) must have the form $[\hat{\boldsymbol{p}}_{\text{TLS}}^T, -1]^T = c\boldsymbol{T}\boldsymbol{v}_3^T$ where $c = -1/(t_3 v_{33})$ and $\boldsymbol{v}_3 = [v_{13}, v_{23}, v_{33}]^T$. Thus, the TLS estimate is given by

$$\hat{\boldsymbol{p}}_{\text{TLS}} = -\frac{1}{t_3 v_{33}}\begin{bmatrix}t_1 v_{13} \\ t_2 v_{23}\end{bmatrix}. \tag{11}$$

4. ERROR STATISTICS FOR $\boldsymbol{A}$ AND $\boldsymbol{b}$

The mean values of the errors between the measured and noise-free entries of $[\boldsymbol{A}, \boldsymbol{b}]$ are

$$E\{\sin\tilde{\theta}_k - \sin\theta_k\} = \bar{\gamma}_k \sin\theta_k \tag{12a}$$
$$E\{-\cos\tilde{\theta}_k + \cos\theta_k\} = -\bar{\gamma}_k \cos\theta_k \tag{12b}$$
$$E\{\boldsymbol{a}_k^T\boldsymbol{r}_k - [\sin\theta_k, -\cos\theta_k]\boldsymbol{r}_k\} = \bar{\gamma}_k[\sin\theta_k, -\cos\theta_k]\boldsymbol{r}_k \tag{12c}$$

where $\bar{\gamma}_k = E\{\cos n_k\} - 1$.

For sufficiently small bearing noise, we have $\sin n_k \approx n_k$ and $\cos n_k \approx 1$, which results in

$$\sin\tilde{\theta}_k - \sin\theta_k \approx n_k \cos\theta_k \tag{13a}$$
$$-\cos\tilde{\theta}_k + \cos\theta_k \approx n_k \sin\theta_k \tag{13b}$$
$$\boldsymbol{a}_k^T\boldsymbol{r}_k - [\sin\theta_k, -\cos\theta_k]\boldsymbol{r}_k \approx n_k[\cos\theta_k, \sin\theta_k]\boldsymbol{r}_k \tag{13c}$$

and

$$E\{(\sin\tilde{\theta}_k - \sin\theta_k)^2\} \approx \sigma_{n_k}^2 \cos^2\theta_k \tag{14a}$$
$$E\{(\cos\tilde{\theta}_k - \cos\theta_k)^2\} \approx \sigma_{n_k}^2 \sin^2\theta_k \tag{14b}$$
$$E\{(\boldsymbol{a}_k^T\boldsymbol{r}_k - [\sin\theta_k, -\cos\theta_k]\boldsymbol{r}_k)^2\} \approx \sigma_{n_k}^2([\cos\theta_k, \sin\theta_k]\boldsymbol{r}_k)^2. \tag{14c}$$

The fact that the noise variance on each entry of $[\boldsymbol{A}, \boldsymbol{b}]$ depends on the true bearing angle as well as the observer positions makes the determination of appropriate diagonal weighting matrices extremely difficult, if not impossible. Therefore, we will assume $\boldsymbol{L} = \boldsymbol{I}$ and $\boldsymbol{T} = \boldsymbol{I}$. As a result, the TLS estimator in (11) becomes

$$\boxed{\hat{\boldsymbol{p}}_{\text{TLS}} = -\frac{1}{v_{33}}\begin{bmatrix}v_{13} \\ v_{23}\end{bmatrix}} \tag{15}$$

where $\boldsymbol{v}_3$ is obtained from the SVD $[\boldsymbol{A}, \boldsymbol{b}] = \sum_{i=1}^{3} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^T$.

5. BIAS OF THE TLS ESTIMATOR

The following theorem gives an expression for the TLS bias in terms of the TLS perturbation $\hat{\boldsymbol{\Delta}}$ and the matrix equation noise $\boldsymbol{\eta}$:

Theorem 1. *For $\boldsymbol{L} = \boldsymbol{I}$ and $\boldsymbol{T} = \boldsymbol{I}$, the bias of the TLS target location estimator is given by*

$$E\{\hat{\boldsymbol{p}}_{TLS}\} - \boldsymbol{p} = -E\{((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T\boldsymbol{A})^{-1}(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T\boldsymbol{\eta}\}. \tag{16}$$

Proof. Using (6), the normal equations for the TLS estimator can be written as

$$(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})\hat{\boldsymbol{p}}_{\text{TLS}} = (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T(\boldsymbol{b} + \hat{\boldsymbol{\delta}}) \tag{17}$$

which can be solved to obtain

$$\hat{\boldsymbol{p}}_{\text{TLS}} = ((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T(\boldsymbol{A} + \hat{\boldsymbol{\Delta}}))^{-1}(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T(\boldsymbol{b} + \hat{\boldsymbol{\delta}}). \tag{18}$$

Using (8) and (9), we have

$$[\boldsymbol{A} + \hat{\boldsymbol{\Delta}}, \boldsymbol{b} + \hat{\boldsymbol{\delta}}]^T[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}] = \begin{bmatrix}(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T\hat{\boldsymbol{\Delta}} & (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T\hat{\boldsymbol{\delta}} \\ (\boldsymbol{b} + \hat{\boldsymbol{\delta}})^T\hat{\boldsymbol{\Delta}} & (\boldsymbol{b} + \hat{\boldsymbol{\delta}})^T\hat{\boldsymbol{\delta}}\end{bmatrix} \tag{19a}$$

$$= -\sigma_3\left(\sum_{i=1}^{2} \sigma_i \boldsymbol{v}_i \boldsymbol{u}_i^T\right)\boldsymbol{u}_3 \boldsymbol{v}_3^T \tag{19b}$$

$$= \boldsymbol{0} \tag{19c}$$

where $\boldsymbol{u}_1^T \boldsymbol{u}_3 = \boldsymbol{u}_2^T \boldsymbol{u}_3 = 0$ since $\boldsymbol{U}$ is an orthogonal matrix. Thus the perturbed augmented matrix $[\boldsymbol{A} + \hat{\boldsymbol{\Delta}}, \boldsymbol{b} + \hat{\boldsymbol{\delta}}]$ and the TLS perturbations $[\hat{\boldsymbol{\Delta}}, \hat{\boldsymbol{\delta}}]$ are orthogonal.

Using (19), (18) can be simplified to

$$\hat{\boldsymbol{p}}_{\text{TLS}} = ((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A})^{-1} (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{b}. \tag{20}$$

Substituting $\boldsymbol{b} = \boldsymbol{A}\boldsymbol{p} - \boldsymbol{\eta}$ (see (3)) into (20) yields

$$\hat{\boldsymbol{p}}_{\text{TLS}} = ((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A})^{-1} (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T (\boldsymbol{A}\boldsymbol{p} - \boldsymbol{\eta}) \tag{21a}$$

$$= \boldsymbol{p} - ((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A})^{-1} (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{\eta}. \tag{21b}$$

Re-arranging and taking the expectation gives (16). □

Asymptotically (as $N \to \infty$) we have

$$\text{plim}\ \hat{\boldsymbol{p}}_{\text{TLS}} = \boldsymbol{p} - \text{plim} \left(\frac{(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A}}{N} \right)^{-1} \text{plim}\ \frac{(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{\eta}}{N} \tag{22}$$

where plim denotes the *probability limit* and is defined by [10]

$$\text{plim}\ \hat{\boldsymbol{p}} = \boldsymbol{p}^* \iff \lim_{N \to \infty} P\{|\hat{\boldsymbol{p}} - \boldsymbol{p}^*| > \epsilon\} = 0$$

for every $\epsilon > 0$. Therefore, for sufficiently large N, the TLS bias can be approximated by

$$-E\{(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A}\}^{-1} E\{(\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{\eta}\}. \tag{23}$$

6. EFFECT OF GEOMETRY SHIFTS ON TLS BIAS

The local Cartesian coordinates can be shifted by $\boldsymbol{\psi}$ by simply adding $\boldsymbol{\psi}$ to a given position vector $\boldsymbol{x}$:

$$\boldsymbol{x}_{\psi} = \boldsymbol{x} + \boldsymbol{\psi}. \tag{24}$$

The TLS estimation error after a coordinate shift is given by

$$\boldsymbol{\beta}'_{\psi} = -((\boldsymbol{A}_{\psi} + \hat{\boldsymbol{\Delta}}_{\psi})^T \boldsymbol{A}_{\psi})^{-1} (\boldsymbol{A}_{\psi} + \hat{\boldsymbol{\Delta}}_{\psi})^T \boldsymbol{\eta} \tag{25a}$$

$$= -((\boldsymbol{A} + \hat{\boldsymbol{\Delta}}_{\psi})^T \boldsymbol{A})^{-1} (\boldsymbol{A} + \hat{\boldsymbol{\Delta}}_{\psi})^T \boldsymbol{\eta}. \tag{25b}$$

Theorem 2. *The TLS estimation error is not invariant to coordinate shifts, i.e.,*

$$\boldsymbol{\beta}'_{\psi} \neq \boldsymbol{\beta}' \text{ for } \boldsymbol{\psi} \neq \boldsymbol{0} \tag{26}$$

where $\boldsymbol{\beta}' = \hat{\boldsymbol{p}}_{TLS} - \boldsymbol{p} = -((\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{A})^{-1} (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})^T \boldsymbol{\eta}$.

Proof. According to (25b), $\boldsymbol{\beta}'_{\psi} = \boldsymbol{\beta}'$ for $\boldsymbol{\psi} \neq \boldsymbol{0}$ if $\hat{\boldsymbol{\Delta}}_{\psi} = \hat{\boldsymbol{\Delta}}$. From (6) we see that if $\hat{\boldsymbol{\Delta}}$ were invariant to coordinate shifts, we would have

$$(\boldsymbol{A} + \hat{\boldsymbol{\Delta}}_{\psi})(\hat{\boldsymbol{p}}_{\text{TLS}} + \boldsymbol{\psi}) = (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})(\hat{\boldsymbol{p}}_{\text{TLS}} + \boldsymbol{\psi}) \tag{27a}$$

$$= \boldsymbol{b} + \hat{\boldsymbol{\delta}} + (\boldsymbol{A} + \hat{\boldsymbol{\Delta}})\boldsymbol{\psi} \tag{27b}$$

$$= \boldsymbol{b}_{\psi} + \hat{\boldsymbol{\delta}}_{\psi} \tag{27c}$$

where $\hat{\boldsymbol{\delta}}_{\psi} = \hat{\boldsymbol{\delta}} + \hat{\boldsymbol{\Delta}}\boldsymbol{\psi}$. Thus, the TLS estimation error invariance to coordinate shifts requires the TLS perturbations to be

$$\hat{\boldsymbol{\Delta}}_{\psi} = \hat{\boldsymbol{\Delta}}, \quad \hat{\boldsymbol{\delta}}_{\psi} = \hat{\boldsymbol{\delta}} + \hat{\boldsymbol{\Delta}}\boldsymbol{\psi}. \tag{28}$$

Given the SVD $[\boldsymbol{A}, \boldsymbol{b}] = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^T$, a coordinate shift by $\boldsymbol{\psi}$ results in

$$[\boldsymbol{A}_{\psi}, \boldsymbol{b}_{\psi}] = [\boldsymbol{A}, \boldsymbol{b} + \boldsymbol{A}\boldsymbol{\psi}] \tag{29a}$$

$$= \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}_{\psi}^T \tag{29b}$$

where

$$\boldsymbol{V}_{\psi} = \begin{bmatrix} \boldsymbol{V}_1 \\ (\boldsymbol{\nu}_2 + \boldsymbol{V}_1^T \boldsymbol{\psi})^T \end{bmatrix} \tag{30}$$

with

$$\boldsymbol{V}_1 = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix}, \quad \boldsymbol{\nu}_2 = \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}. \tag{31}$$

Equation (29b), which assumes the hypothesized forms of the TLS perturbations in (28), cannot be an SVD of $[\boldsymbol{A}_{\psi}, \boldsymbol{b}_{\psi}]$ since $\boldsymbol{V}_{\psi}^T \boldsymbol{V}_{\psi} \neq \boldsymbol{I}$, i.e., $\boldsymbol{V}_{\psi}$ is not an orthogonal matrix. Thus, the TLS perturbations given by (28) that are required to enable the invariance of the TLS estimation error to coordinate shifts are not tenable. In other words, $\hat{\boldsymbol{\Delta}}_{\psi} \neq \hat{\boldsymbol{\Delta}}$ which implies $\boldsymbol{\beta}'_{\psi} \neq \boldsymbol{\beta}'$ for $\boldsymbol{\psi} \neq \boldsymbol{0}$. □

The optimal coordinate shift for minimizing the TLS bias is

$$\boldsymbol{\psi}_{\text{min}} = \arg\min_{\boldsymbol{\psi}} \|E\{\boldsymbol{\beta}'_{\psi}\}\|_2. \tag{32}$$

A closed-form solution for the above minimization problem is not available. Referring to (13), it is seen that the errors in $\boldsymbol{A}$ are invariant to coordinate shifts while those in $\boldsymbol{b}$ *are* dependent on coordinate shifts. If the localization coordinates are shifted away from the origin, this has the effect of increasing the observer location vector norms, which in turn increases the second-order moments of error on $\boldsymbol{b}$. This unproportionate increase in the second-order moments tends to deteriorate the TLS bias.

To reduce the TLS bias requires to shift the localization coordinates towards the origin so as to reduce the error on $\boldsymbol{b}$. Using this observation, the TLS bias may be reduced by choosing a coordinate shift that minimizes the sum of second-order moments for the errors in $\boldsymbol{b}$:

$$\min_{\boldsymbol{\psi}} \sum_{k=1}^{N} \sigma_{n_k}^2 ([\cos\theta_k, \sin\theta_k](\boldsymbol{r}_k + \boldsymbol{\psi}))^2. \tag{33}$$

The above minimization problem does not generally yield minimum-bias coordinate shifts although it may enable significant bias reduction.

In target localization problems, only measured noisy bearing angles are available. Even though (33) makes use of noise-free bearings θ_k, a good estimate for the bias-minimizing shift $\boldsymbol{\psi}_{\text{min}}$ can be obtained by replacing the θ_k with the $\tilde{\theta}_k$. While this modification yields a practical criterion for selecting $\boldsymbol{\psi}$, its accuracy will depend on the bearing noise variance among other things.

7. SIMULATION STUDIES

The original simulated target localization geometry is shown in Fig. 2 where the true target location is $\boldsymbol{p} = [-17.4, 98.5]^T$ km and $N = 40$ bearing measurements are taken at regular intervals along a linear trajectory between $\boldsymbol{r}_1 = [0, 30]^T$ km and $\boldsymbol{r}_N = [40, 20]^T$ km. The bearing noise standard deviation is 5°. As is evident from Fig. 2, both the pseudolinear and TLS estimators exhibit large estimation bias even though the TLS bias is smaller.

To demonstrate the effect of local coordinate shifts on the TLS bias, the TLS bias was estimated for each shift in the range $-40 \leq \psi_x \leq 0$ and $-40 \leq \psi_y \leq -10$. The resulting TLS bias norm surface is plotted in Fig. 3. The bias surface is not flat and appears to favour certain coordinate shifts as far as bias reduction is concerned. The minimum bias was obtained at shift $\boldsymbol{\psi}_{\text{min}} = [-24, -23]^T$. The geometry for this shift is shown in Fig. 4 along with the pseudolinear and TLS estimation results. A comparison of Figs. 2 and 4 confirms that the TLS estimation bias is significantly reduced by shifting the

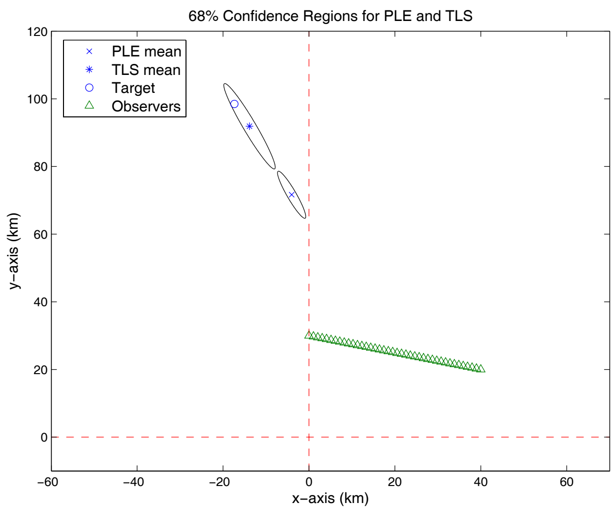

Fig. 2. Simulated geometry with error ellipses for PLE and TLS estimators.

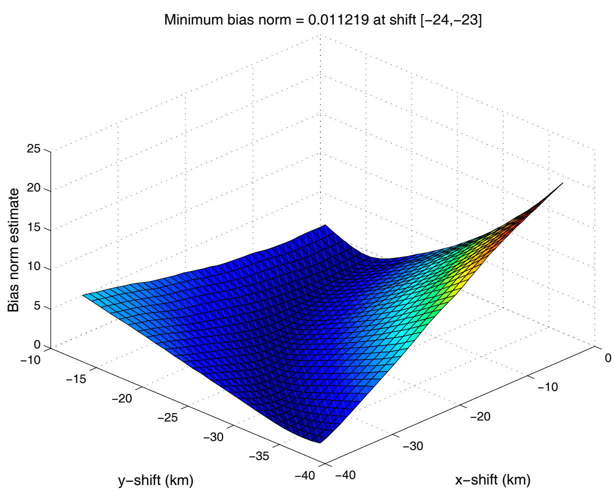

Fig. 3. TLS estimation bias as a function of coordinate shift.

origin of the local coordinates in Fig. 2 by $[-24, -23]^T$ km while the PLE bias remains invariant to coordinate shifts. As predicted in Section 6, bringing the receiver locations close to the origin appears to do the trick when it comes to reducing the TLS estimation bias.

8. CONCLUSIONS

This paper has analyzed the sensitivity of the TLS estimation bias to local coordinate shifts in bearings-only target localization problems. Optimal coordinate shift that gives the minimum bias is not easy to determine without resorting to computer simulations. It was discovered that the observers must be close to the origin of the local coordinates after a coordinate shift in order to ensure some bias reduction.

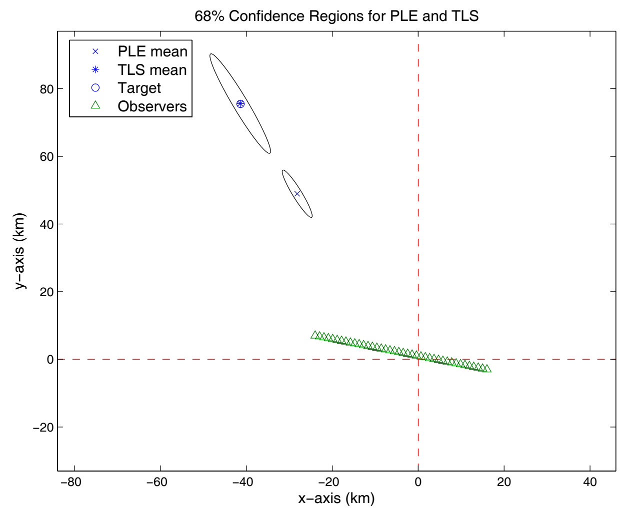

Fig. 4. Minimum-bias geometry and error ellipses for PLE and TLS estimators.

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