THE PEP APPROACH: A NEW FAMILY OF METHODS SOLVING THE PHASE ESTIMATION PROBLEM

Amar Kachenoura, Laurent Albera and Lotfi Senhadji

INSERM UMR 642, Laboratoire Traitement du Signal et de l'Image, Rennes, France
Université de Rennes 1, LTSI, Campus de Beaulieu, F-35042 Rennes, France

ABSTRACT

Knowledge of the q-th ($q>2$) order spectrum of a linear non-Gaussian process allows to reconstruct both the magnitude and the phase of the corresponding input sequence. We propose in this paper a new family of phase retrieval algorithms, based on higher order spectra, named PEP (Phase Estimation using Polyspectra). These new algorithms are easier to implement and use. Moreover, computer simulations show that among them, the 4-PEP and the (3,4)-PEP algorithms exhibit good performances facing classical methods especially for bandlimited systems.

1. INTRODUCTION

System reconstruction and especially phase recovery play an important role in various application areas. For instance, in astronomy, high resolution imaging from ground-based telescopes involves a phase recovery in order to overcome the severe atmospheric degradation. Moreover, the phase estimation problem also appears in radiocommunications, speech processing and medical diagnosis and more particularly for the use of blind source deconvolution based on frequency-domain [1].

In this paper, we focus on the phase retrieval problem of Single-Input Single-Output (SISO) systems whose input is an i.i.d. non-Gaussian process. More particularly, we are interested in the nonparametric methods. They can be divided in two subcategories: those that utilize the whole bispectrum (or polyspectrum) information [2] [3] and those that use only some part of the polyspectrum [4] such as one or two fixed One-Dimensional (1D) polyspectrum slices [5] [6] [7]. Although some of the entire bispectrum methods [2] [3] provide the additional option of using a subset of the bispectrum, none of them supply a procedure for selecting the most useful (in the sense of improved system estimates) bispectrum information. On the contrary, the algorithm of [6] proposes such a procedure, wich can be applied to [5] too. This selection procedure potentially allows one to avoid regions where polyspectrum estimates exhibit high variance or regions where the ideal polyspectrum is expected to be zero, such as in the case of bandlimited systems.

Each of these methods suffers from limitations. To start with, the methods that use the whole polyspectrum information [2] [3] are generally more sensitive when systems are bandlimited as shown in [6]. Besides, Rangoussi et al. [3] and Lii et al. [7] have developed algorithms only for real systems. The algorithms proposed in [5] and [6] reconstruct the phase only up to a linear-phase component corresponding to an integer time delay. Besides, the method given in [5] does not allow to process a linear process whose input sequence is symmetrically distributed.

In order to overcome the limitations of the previous algorithms, a new family of phase retrieval methods, named PEP (Phase Estimation using Polyspectra), is proposed in this paper. This latter, based on higher order spectra (polyspectra), offers a panel of algorithms which are more straightforward to implement and use. Moreover some methods of the PEP family, such as the 4-PEP and (3,4)-PEP algorithms are more robust in the case of bandlimited systems than existing methods.

2. NOTATIONS AND DATA STATISTICS

2.1. Problem Formulation

It is assumed throughout the paper that M (a priori complex) samples of a discrete stochastic process are observed, and that each random variable $x(m)$ of the process satisfies the following Linear Time Invariant (LTI) model:

$$x(m)=\sum_{\ell\,\in\,\mathbb{Z}} h(\ell)\, s(m-\ell)+\nu(m)=(h*s)(m)+\nu(m) \quad (1)$$

where $\{s(m)\}_{m\in\mathbb{Z}}$ and $\{\nu(m)\}_{m\in\mathbb{Z}}$ represent the input and additive noise processes respectively, and where

$$h(\ell) \overset{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathcal{H}(\omega)\ \mathrm{e}^{\,\mathrm{i}\,\omega\ell}\, d\omega \quad (2)$$

is the impulse response of the LTI system. Note that $\mathcal{H}(\omega)$ denotes its frequency transfer function.

Moreover, the following assumptions are placed on the system and the signals involved.

A1. $\{s(m)\}_{m\in\mathbb{Z}}$ is an i.i.d. non-Gaussian, stationary and ergodic process with components a priori in the complex field;

A2. Noise $\{\nu(m)\}_{m\in\mathbb{Z}}$ is stationary, ergodic, Gaussian with components a priori in the complex field too, and independent of the input process;

A3. All q-th ($q > 2$) order marginal source cumulants are absolutely summable and all q-th order spectra are nonzero in the frequency band over which the channel response is nonzero (higher order cumulants and polyspectra will be described in section 2.2);

A4. The LTI system is stable (i.e. $\{h(m)\}_{m\in\mathbb{Z}}$ is absolutely summable, which guarantees the existence of a bounded frequency response) with a priori complex taps.

If we express:

$$\mathcal{H}(\omega) = |\mathcal{H}(\omega)|\ \mathrm{e}^{\,\mathrm{i}\phi_h(\omega)} \tag{3}$$

the phase retrieval problem is to blindly reconstruct the phase response, $\phi_h(\omega)$, of the LTI system, namely only from output samples $x(m)$.

2.2. Cumulants and spectra of q-th ($q\geq 3$) order

Denote as $C^{q-r}_{r,\,x}(m, \tau_1, \ldots, \tau_{q-1})$ the q-th ($q \geq 3$) order cumulant of $x(m)$, defined by:

$$C^{q-r}_{r,\,x}(m, \tau_1, \ldots, \tau_{q-1}) = \mathsf{Cum}\{x(m), x(m+\tau_1), \ldots, x(m+\tau_{r-1}), x(m+\tau_r)^*, \ldots, x(m+\tau_q)^*\} \tag{4}$$

where r terms are not conjugated and $q-r$ terms are conjugated. Note that in the presence of stationary sources, q-th order cumulants do not depend on time m, so they can be denoted by $C^{q-r}_{r,\,x}(\tau_1, \ldots, \tau_{q-1})$. Nevertheless, if sources are cyclostationary, cycloergodic, potentially non zero-mean, q-th order continuous-time temporal mean statistics have to be used instead of (4), as described in [8]. For the sake of convenience, we will consider only the stationary case in the sequel as announced by assumptions (**A1**) and (**A2**).

Under assumption (**A3**), the q-th order spectrum (polyspectrum) is given by the $(q-1)$-dimensional Discrete Fourier Transform (DFT) of the q-th order cumulant [4]. Besides, it can be obviously shown using equation (4), assumptions (**A1**), (**A2**) and the multilinearity property under changes of systems, shared by all moments and cumulants, that the q-th order spectrum of the output $x(m)$ may be written as following:

$$\Gamma^{q-r}_{r,\,x}(\omega_1, \ldots, \omega_{q-1}) = C^{q-r}_{r,\,s}\,\mathcal{H}(-\omega_1 - \ldots - \omega_{q-1})\, \mathcal{H}(\omega_1)\ldots\mathcal{H}(\omega_{r-1})\,\mathcal{H}(-\omega_r)^*\ldots\mathcal{H}(-\omega_{q-1})^* \tag{5}$$

where $C^{q-r}_{r,\,s} \overset{\text{def}}{=} C^{q-r}_{r,\,s}(0, \ldots, 0)$ denotes the q-th order marginal source cumulant associated with null delays.

Generally, using the well-known Leonov-Shiryaev formula, q-th order cumulants (4) are computed from moments of order smaller than or equal to q. [8] illustrates the Leonov-Shiryaev formula for $q = 4$ and $q = 6$. However, in practice, moments and cumulants can not be exactly computed; they have to be estimated from samples $x(m)$. If the sources are stationary and ergodic, sample statistics may be used to estimate moments, and consequently to estimate cumulants (4), via the Leonov-Shiryaev formula. Next, polyspetrum estimation is achieved using the DFT (see [4] for details).

3. ALGORITHMS

We present in this section a new family of algorithms, the PEP (Phase Estimation using Polyspectra) family, which allows to reconstruct the phase response of the system up to an additive constant. This family includes, on the one hand, the q-PEP methods, which exploits the q-th order spectrum ($q\geq 3$), and, on the other hand, the (q_1, q_2)-PEP methods, based both on q_1-th and q_2-th order spectra ($q_2 > q_1 \geq 3$).

3.1. The q-PEP methods ($q\geq 3$)

The approach is presented using as example the spectrum of third order ($q=3$), well known as *bispectrum*. An extension to q-th order ($q>3$) spectra is straightforward and thus omitted for the sake of convenience.

For $q=3$ and $r=2$, equation (5) obviously becomes:

$$\Gamma^{1}_{2,\,x}(\omega_1, \omega_2) = C^{1}_{2,\,s}\ \mathcal{H}(-\omega_1 - \omega_2)\,\mathcal{H}(\omega_1)\,\mathcal{H}(-\omega_2)^* \tag{6}$$

In equation (3) we have defined the phase response, $\phi_h(\omega)$, of the LTI system. Now let $\psi^{1}_{2,\,x}(\omega_1, \omega_2)$ be the phase of the output bispectrum $\Gamma^{1}_{2,\,x}(\omega_1, \omega_2)$. In the sequel, we consider discrete frequencies, i.e., $\omega_i = (2\pi/N)k_i$ with $k_i \in \{0, \ldots, N-1\}$. Then, omitting factor $2\pi/N$, the relation between the phases of the quantities involved in (6) can be written as:

$$\psi^{1}_{2,\,x}(k_1, k_2) = \phi_h(-k_1 - k_2) + \phi_h(k_1) - \phi_h(-k_2) + \xi^{1}_{2,\,s} \tag{7}$$

where $\xi^{1}_{2,\,s}$ is the phase associated with the marginal source cumulant $C^{1}_{2,\,s}$. Note that for a source in the real field, $\xi^{1}_{2,\,s}$ is a multiple of π. Moreover, the 2π-periodicity of $\mathcal{H}(\omega)$ implies the N-periodicity of its discrete phase $\phi_h(k)$. So, summing (7) over the discrete frequencies k_2 ($0\leq k_2 < N$), we have for each discrete frequency k_1 ($0\leq k_1 < N$):

$$\sum_{k_2=0}^{N-1} \psi^{1}_{2,\,x}(k_1, k_2) = N\big(\phi_h(k_1) + \xi^{1}_{2,\,s}\big) \tag{8}$$

which implies that $\phi_h(k)$ can be computed from the bispectrum phase. However, even if equation (8) provides a solution for $\phi_h(k_1)$ from $\psi^{1}_{2,\,x}(k_1, k_2)$, it is not a convenient formula for phase retrieval. Indeed, the bispectrum phase $\psi^{1}_{2,\,x}(k_1, k_2)$ is generally estimated by its principal value, $\tilde{\psi}^{1}_{2,\,x}(k_1, k_2) = \arctan\big(\Im\big(\Gamma^{1}_{2,\,x}(k_1, k_2)\big)\,, \Re\big(\Gamma^{1}_{2,\,x}(k_1, k_2)\big)\big)$ where $\Re$ and $\Im$ refer to the real and imaginary parts, and $\arctan$ is the four-quadrant arc tangent operator where angles $\tilde{\psi}^{1}_{2,\,x}(k_1, k_2)$ ($0 \leq k_1, k_2 < N$) lie between $\pm\pi$ radians. These principal values are also called *wrapped* phase values because the absolute phase is wrapped into the interval $]-\pi, \pi\,]$ by the following nonlinear process:

$$\tilde{\psi}^{1}_{2,\,x}(k_1, k_2) = \psi^{1}_{2,\,x}(k_1, k_2) + 2\pi I(k_1, k_2) \tag{9}$$

where $I(k_1, k_2)$ is an integer function such that $\tilde{\psi}^{1}_{2,\,x}(k_1, k_2)$ belongs to $]-\pi, \pi\,]$. Thus, summing (9) over the discrete frequencies k_2 ($0 \leq k_2 < N$) and using (8), we have:

$$\sum_{k_2=0}^{N-1} \tilde{\psi}_{2,x}^{1}(k_1, k_2) = N\big(\phi_h(k_1) + \xi_{2,s}^{1}\big) + 2\pi J(k_1) \quad (10)$$

where the integer function $J(k_1)$ is given, for $0 \leq k_1 < N$, by $J(k_1) \stackrel{\text{def}}{=} \sum_{k_2=0}^{N-1} I(k_1, k_2)$. The phase function $\phi_h(k)$ can thus be extracted from equation (10). Nevertheless, phase *unwrapping* has to be achieved before extraction. Simply stated, the phase unwrapping problem is concerned with obtaining an estimate for a continuous function from its wrapped form. Applying a Two-Dimensional (2D) phase unwrapping scheme [9] to $\tilde{\psi}_{2,x}^{1}(k_1, k_2)$ allows to determine an estimate, $\tilde{\psi}_{2,x}^{1,u}(k_1, k_2)$, of $\psi_{2,x}^{1}(k_1, k_2)$ up to an additive constant such that:

$$\sum_{k_2=0}^{N-1} \tilde{\psi}_{2,x}^{1,u}(k_1, k_2) = N\big(\phi_h(k_1) + \xi_{2,s}^{1} + 2\pi I_u\big) \quad (11)$$

where I_u is an unknown integer constant. An estimate, $\tilde{\phi}_h(k)$, of $\phi_h(k)$ can thus be achieved using the following scheme:

$$\tilde{\phi}_h(k_1) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{k_2=0}^{N-1} \tilde{\psi}_{2,x}^{1,u}(k_1, k_2) = \phi_h(k_1) + a \quad (12)$$

where constant a is given by $a = \xi_{2,s}^{1} + 2\pi I_u$. This approach will be referred, in the sequel, to as the 3-PEP$_{2D}$ method. On the other hand, a One-Dimensional (1D) phase unwrapping process may be sufficient to solve our problem. Indeed, applying it to the left term of equation (10) and dividing the result by N, we obtain a new estimate, $\tilde{\phi}_h(k)$, of $\phi_h(k)$ given by (12) where now $a = \xi_{2,s}^{1} + (2\pi J_u)/N$ and J_u is an unknown constant. This latter approach will be referred in the sequel to as the 3-PEP$_{1D}$ algorithm. Then the 3-PEP concept may be easily extended to q-th order spectra ($q > 3$), which is necessary, for instance, in the presence of a symmetrically distributed input. In addition, note that the q-PEP method, for $q > 3$, exploits only one 2D-slice of the output q-th order spectrum. A simple measure of 2D-slice "goodness" can be derived from the one presented in [6] and referred to as the *frequency content*. On the other hand, since a q-th order spectrum ($q > 3$) may contain several 2D-slices of sufficient goodness, an improved final phase estimate can be obtained by averaging. Moreover, this averaging should be done in the $e^{i\cdot}$ domain just before the division by N.

3.2. The (q_1, q_2)-PEP methods ($q_2 > q_1 \geq 3$)

The originality of this algorithm is the joint exploitation of the 2D-slice of two polyspectra of different order. As example, the method is presented in this section using the third ($q_1 = 3$) and the fourth ($q_2 = 4$) order spectra of the observations. The extension to (q_1, q_2)-th order, such as $q_2 > q_1 \geq 3$, can also be easily realized from the following discussion.

For $(q_2, r_2) = (4, 2)$, the equation (5) implies:

$$\psi_{2,x}^{2}(k_1, k_2, -k_3) = \phi_h(-k_1 - k_2 + k_3) + \phi_h(k_1) - \phi_h(-k_2) - \phi_h(k_3) + \xi_{2,s}^{2} \quad (13)$$

where $\psi_{2,x}^{2}(k_1, k_2, -k_3)$ is the phase of the discrete output trispectrum $\Gamma_{2,x}^{2}(k_1, k_2, -k_3)$ and $\xi_{2,s}^{2}$ is the phase associated with the marginal source cumulant $C_{2,s}^{2}$. Based on the difference between equation (7) and equation (13), we get:

$$\psi_{2,x}^{1}(k_1, k_2) - \psi_{2,x}^{2}(k_1, k_2, -k_3) = \xi_{2,s}^{1} - \xi_{2,s}^{2} - \phi_h(-k_1 - k_2 + k_3) + \phi_h(-k_1 - k_2) + \phi_h(+k_3). \quad (14)$$

Next, k_1 has to be fixed, using for instance the frequency content [6], to a given frequency α ($0 \leq \alpha < N$). Summing up (14) over all discrete frequencies k_2 ($0 \leq k_2 < N$), we get:

$$\sum_{k_2=0}^{N-1} \big(\psi_{2,x}^{1}(\alpha, k_2) - \psi_{2,x}^{2}(\alpha, k_2, -k_3)\big) = N\big(\phi_h(k_3) - \xi_{2,s}^{2} + \xi_{2,s}^{1}\big) \quad (15)$$

Therefore, the phase response, $\phi_h(k)$, can be estimated by the joint exploitation of the 2D-slice of two polyspectra of different order. However, as shown in the previous section, the output polyspectrum phases are estimated through their principal values. To obtain the true phases up to an additive constant, we must perform an additional step of phase unwrapping. This problem can be resolved at least in two different ways. The first one is to applying a 1D phase unwrapping to $\sum_{k_2=0}^{N-1}\Big(\tilde{\psi}_{2,x}^{1}(\alpha, k_2) - \tilde{\psi}_{2,x}^{2}(\alpha, k_2, -k_3)\Big)$. This method will be referred to as the $(3,4)$-PEP$_{1D}$ algorithm. The second one, is to applying a 2D phase unwrapping to $\tilde{\psi}_{2,x}^{2}(\alpha, k_2, -k_3)$ before summing over all discrete frequencies k_2. This approach will be named the $(3,4)$-PEP$_{2D}$ method in the sequel.

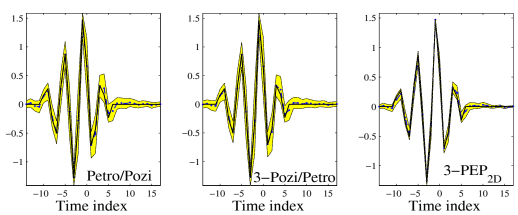
4. COMPUTER SIMULATIONS

Two computer experiments show the performances of the PEP family (more particularly through the 3-PEP$_{2D}$, 4-PEP$_{2D}$ and $(3,4)$-PEP$_{2D}$ methods) and some efficient phase retrieval techniques (Petro/Pozi [5], 3-Pozi/Petro [6] using the data bispectrum and 4-Pozi/Petro [6] using the data trispectrum) for bandlimited systems. Note that Pozidis and Petropulu have demonstrated in [6] via simulations the superiority, in terms of estimation bias and variance, of the Pozi/Petro methods over the approaches proposed in [2] and [3] in the case of bandlimited systems. So we did not implement these two latters. In fact, the input source used in the first experiment was a stochastic process with zero-mean i.i.d. exponentially distributed random variables whereas, in the second experiment, it was a Binary Phase Shift Keying (BPSK) source in baseband with a square transmit filter and a symbol rate equal to the sample rate. Consequently, since the i.i.d. exponential process has a non zero skewness, the first experiment allows to compare the performances of the 3-PEP$_{2D}$, (3,4)-PEP$_{2D}$, Petro/Pozi and the 3-Pozi/Petro algorithms. On the other hand, since the BPSK is symmetrically distributed, the trispectrum of the observation was used in the second experiment permitting to compare the performances of the 4-PEP$_{2D}$ and the 4-Pozi/Petro methods. Besides, in both experiments, we generated a nonminimum-phase, bandlimited filter whose impulse response is given by:

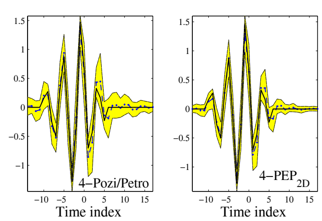
$$h(\ell) = 0.77^{|2\ell|}\cos(1.96\pi\ell) + 0.8(0.65)^{|2\ell|}\sin(1.52\pi\ell + \frac{\pi}{5})$$

and the input process had an SNR (Signal to Noise Ratio) of 15 dB. Eventually, the simulation results are averaged over 200 realizations.

Figure 1 displays, for a number of 1024 samples, the true impulse response (solid line) of the bandlimited filter, along with the mean of the impulse response (dotted line) recovered by five methods (namely Petro/Pozi, 3-Pozi/Petro, 3-PEP$_{2D}$, 3-Pozi/Petro and 4-PEP$_{2D}$). The yellow (or gray for a black and white printing) area denotes the standard deviation. Figure 1(a) presents the simulation results associated with the first experiment whereas figure 1(b) shows those corresponding to the second experiment. Note that the recovered impulse responses were computed in time-domain from the true filter magnitude combined with the recovered phase. Moreover, the frequency content presented in [6] was previously performed for each method in order to select the suitable 1D or 2D-slice. Figure 1 shows the good performances of the PEP family facing the other methods. Indeed, it can be seen that the PEP methods perform better especially in terms of variance.



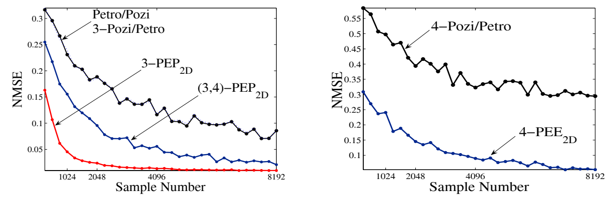
(a) For an exponential input distribution.



(b) For a symmetrical input distribution.

Fig. 1. Phase recovery for 1024 samples and a SNR of 15dB.

Figures 2(a) and 2(b) display the variations of the NMSE (Normalized Mean Square Error) criterion at the output of the six methods (namely the previous ones and the $(3{,}4)$-PEP$_{2D}$ method) as a function of the number of samples, corresponding to both experiments respectively. In both cases, the fast decay of NMSE can be observed for the PEP methods.



(a) For an exponential distribution. (b) For a symmetrical distribution.

Fig. 2. Variations of NMSE as function of the sample number.

5. CONCLUSION

We presented in this paper a new family of phase retrieval algorithms named the q-PEP ($q \geq 3$) and (q_1, q_2)-PEP ($q_2 > q_1 \geq 3$) methods. These algorithms use one or two $2D$-slices of polyspectra of the output observations. The computer results show the good performances of this new class of methods in the presence of band limited system, especially facing classical algorithms. Moreover, inside the PEP family, the q-PEP methods seem to perform better than the (q_1, q_2)-PEP methods. However, the works in progress show that the (q_1, q_2)-PEP methods are less sensitive to the choice of the $2D$-slice.

6. REFERENCES

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