DESIGN OF OVERSAMPLING $\Delta\Sigma$ DA CONVERTERS VIA H^∞ OPTIMIZATION

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ABSTRACT

In this paper, we propose a new method for designing oversampling $\Delta\Sigma$ DA converters via H^∞ optimization. The design consists of two steps. One is that for $\Delta\Sigma$ modulators. In $\Delta\Sigma$ modulators, the accumulator $1/(z-1)$ is conventionally used in a feedback loop to shape quantization noise. In contrast, we give all stabilizing feedback filters for the modulator, and propose an H^∞ design to shape the frequency response of the noise transfer function (NTF). The other is a design for interpolation filters in oversampling DA converters. While conventional designs are executed in the discrete-time domain, we take account of the characteristic of the original analog signal by using sampled-data H^∞ optimization. A design example is presented to show that our design is superior to conventional ones.

1. INTRODUCTION

$\Delta\Sigma$ *modulators* are widely used in high-resolution AD or DA converters. They are applied to measurement, digital audio processing and wireless communications (see [1, 2]). In combination with oversampling technique, $\Delta\Sigma$ AD or DA converters can have high resolution despite a coarse (by usual one-bit) quantizer.

$\Delta\Sigma$ modulators reduce quantization noise by linear filters in feedback loops, which are designed to shape the frequency characteristic of the noise transfer function (NTF). The design is commonly done by assuming that the quantization noise is white, and independent of the input signal. Although the assumption is not strictly valid, the method leads to a linear model. We can then adopt linear system theory, in particular, frequency domain approach. Noise shaping in the frequency domain can be executed by the established H^∞ optimization, and hence such a linear model will be useful to design $\Delta\Sigma$ modulators. Moreover, attenuating the H^∞ norm of the NTF leads the stability of binary (1-bit) (*Lee criterion*[3, 2]) and multi-bit (see section 2.3) modulators.

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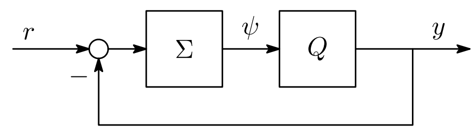


Fig. 1. $\Delta\Sigma$ modulator

Then, we propose a new design method of $\Delta\Sigma$ DA converters. Since DA converters involve both continuous- and discrete-time signals, it is necessary for analysis and design to take the characteristics of both of them into account. For this purpose, the *sampled-data control* [4] is an optimal tool. In the last few years, several studies have been made on digital signal processing via sampled-data control theory [5, 6]. Based on these studies, we propose sampled-data H^∞ optimization for designing $\Delta\Sigma$ converters. By sampled-data H^∞ optimization, we can optimize intersample response of the signals in $\Delta\Sigma$ converters, while only sampled values are optimized by conventional methods. We present design examples to show that our design is superior to conventional ones.

2. $\Delta\Sigma$ MODULATORS

2.1. Conventional modulators

Fig. 1 shows the block diagram of a conventional $\Delta\Sigma$ modulator. In this figure, the difference between the input r and the output y is fed back to Σ, which is conventionally an accumulator $\Sigma(z) = 1/(z-1)$ and outputs a signal ψ. Then the signal ψ is quantized by a quantizer Q, which is a piecewise constant function $\mathbb{R} \to \mathcal{Q}$ where $\mathcal{Q}$ is a finite subset of $\mathbb{R}$.

The quantizer Q is a nonlinear system. To make the analysis easy, we introduce a linear model for Q. Define the quantization error n, that is, $n := Q(\psi) - \psi$. Assuming that the error n is independent of the input ψ, we take the additive noise model for the quantizer as shown in Fig. 2. By using this model, we see that the input-output equation of the conventional modulator is obtained by

$$y = \frac{\Sigma}{1+\Sigma} r + \frac{1}{1+\Sigma} n = z^{-1} r + (1 - z^{-1}) n.$$

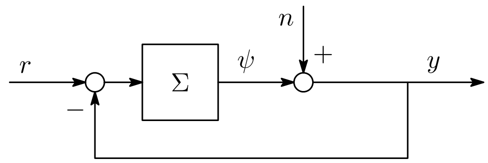


Fig. 2. Linear model for $\Delta\Sigma$ modulator

Since $1/(1+\Sigma) = 1 - z^{-1}$ is high-pass, the quantization noise is reduced at low frequencies and increased at high frequencies. If the input signal r contains few high frequency components, we can separate the noise n from the output signal y by an appropriate lowpass filter.

Therefore, the accumulator Σ plays a noise-shaping role in $\Delta\Sigma$ modulators. In the next section, we generalize this property.

2.2. Quantization noise shaping

We here consider the linear model in Fig. 2. Let $\mathcal{S}$ denote the family of all stable, causal, real-rational transfer functions and

$$\mathcal{S}' := \{G \in \mathcal{S} : G \text{ is strictly causal}\}.$$

Then we characterize Σ for the linear system in Fig. 2.

Lemma 1. *The linearized feedback system Fig. 2 is well-posed and internally stable if and only if*

$$\Sigma \in \left\{ \frac{R}{1-R} : R \in \mathcal{S}' \right\}$$

Proof. See chapter 5 in [7]. □

This theorem gives all stabilizing feedback filters. By using the parameter $R \in \mathcal{S}'$, the input/output relation of the system Fig. 2 is given by

$$y = Rr + (1-R)n =: Rr + Hn, \qquad (1)$$

where $H = 1 - R$ is the noise transfer function (NTF) to be designed. Note that the conventional first order $\Delta\Sigma$ modulator has $R(z) = z^{-1} \in \mathcal{S}'$.

In implementation, finite-impulse response (FIR) filters are often preferred, and hence we assume R is an FIR filter (so is the NTF H), that is,

$$R(z) = \sum_{k=1}^{N} a_k z^{-k}, \quad H(z) = 1 - \sum_{k=1}^{N} a_k z^{-k}$$

Note that $R(z)$ is always in $\mathcal{S}'$. If a desired NTF $H_{\text{des}}(z)$ is given by an FIR filter, $R(z)$ is obtained by $R(z) = 1 - H_{\text{des}}(z)$. Since $R(z)$ must be strictly causal, we have to restrict $H_{\text{des}}(\infty) = 1$.

On the other hand, if $H_{\text{des}}(z)$ is given by an IIR filter, our problem is to approximate $H_{\text{des}}(z)$ by an FIR filter $H(z)$. Since desired NTFs are given by their frequency characteristics, approximation of $H(z)$ should be done in the frequency domain. Therefore, we formulate our problem as an H^∞ optimization:

Problem 1. *Given a stable transfer function $H_{\text{des}}(z)$ (desired NTF) and a stable weighting function $W(z)$, find $H \in \mathcal{S}$ with $H(\infty) = 1$ which minimize $\|(H - H_{\text{des}})W\|_\infty$.*

Then the optimization is reducible to a linear matrix inequality (LMI) with respect to a matrix variable and the coefficients $a_1, \ldots, a_N$ [8], and can be effectively solved by standard optimization software (e.g., MATLAB).

Moreover, the zeros of $H(z)$ can be assigned by linear equations (linear constraints) of $a_1, \ldots, a_N$. Define $n_H(z) := z^N - \sum_{k=1}^{N} a_k z^{N-k}$. Then, $H(z)$ has M zeros at $z = z_0$ if and only if

$$\left. \frac{d^k n_H(z)}{dz^k} \right|_{z=z_0} = 0, \quad k = 0, 1, \ldots, M-1,$$

where $\frac{d^0 n_H(z)}{dz^0} := n_H(z)$. The LMI with these linear constraints can be also effectively solved.

2.3. Stability Constraints

The linearized model Fig. 2 is useful for analyzing the noise shaping properties of $\Delta\Sigma$ modulators. The stability of $\Delta\Sigma$ modulators, however, should be analyzed by considering their nonlinear behaviors.

To analyze the stability, the H^∞ norm of $H(z)$ (NTF) is available. For the stability of binary $\Delta\Sigma$ modulators, the following criterion (Lee criterion) is widely used [3, 2]:

$$\|H\|_\infty < 1.5. \qquad (2)$$

Note that this is not a sufficient nor necessary condition for the stability. For multi-bit modulators with M-step quantizer, the following is a sufficient condition for the stability [9, 2]:

$$\|h\|_1 \le M + 2 - \|r\|_\infty, \qquad (3)$$

where $\|h\|_1$ is the ℓ^1 norm of the impulse response of $H(z)$. Let ν denote the order of $H(z)$. Then, we have the following relation [10]:

$$\|h\|_1 \le (2\nu + 1)\|H\|_\infty.$$

By combining this with (3), we have another stability condition.

$$\|H\|_\infty \le \frac{1}{2\nu+1}(M + 2 - \|r\|_\infty). \qquad (4)$$

From the conditions (2) and (4), attenuation of $\|H\|_\infty$ helps the stability. Therefore, we add the following stability constraints to the design of modulators:

$$\|H\|_\infty < C,$$

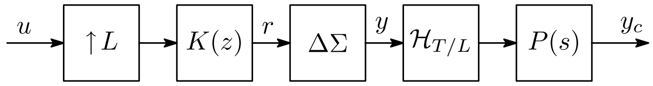


Fig. 3. Oversampling $\Delta\Sigma$ DA converter

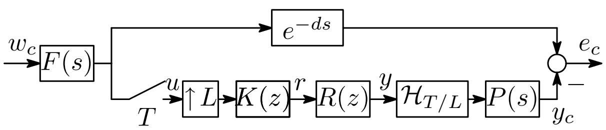


Fig. 4. Error system for designing DA converter

where $C > 0$ is a constant (e.g., by Lee criterion, $C = 1.5$). This inequality is also reducible to LMI [8] and easily combined with the LMI optimization mentioned above.

3. DESIGN OF $\Delta\Sigma$ DA CONVERTERS

By using the feedback filter $R(z)$ characterized in Lemma 1, we here design DA converters with $\Delta\Sigma$ modulators. To take account of intersample response, we introduce sampled-data H^∞ optimization. From the input/output relation (1), the system from r to y is $R(z)$. In this section, we assume that $R(z)$ is given.

3.1. Design problem of $\Delta\Sigma$ DA converters

Fig. 3 shows an oversampling $\Delta\Sigma$ DA (digital-to-analog) converter. Assume that the input signal u has sampling time T and word length b [bits]. The digital signal u is first upsampled by an *interpolator* [11] $K(z)(\uparrow L)$. By $\uparrow L$, $L-1$ zeros are introduced between two consecutive input values [11]. The following digital filter $K(z)$, called *interpolation filter*, operates on the $L-1$ zero-valued samples inserted by $\uparrow L$ to yield nonzero values between the original samples.

Then the interpolated signal r goes through a $\Delta\Sigma$ modulator, and becomes a signal y whose word length is converted to another one, by usual 1 [bit]. Then the discrete-time signal y is converted to a continuous signal by the zero-order hold $\mathcal{H}_{T/L}$ with hold time T/L, smoothed by a continuous-time filter $P(s)$, and finally becomes an analog signal y_c.

Our objective here is to design the interpolation filter $K(z)$ to interpolate samples taking account of the analog performance. If we a priori have the knowledge about the characteristic $F(s)$ of the original analog signal (e.g., u is a sampled data of an orchestral music), we can use it explicitly for design.

Therefore, we consider the error system Fig. 4 for designing the filter $K(z)$. Let $\mathcal{E}$ denote the input/output operator from w_c to e_c. Then, our design problem is then as follows:

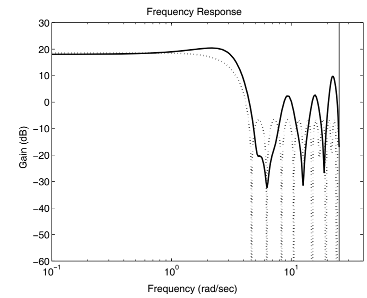


Fig. 5. Interpolation filters for DA converter: sampled-data H^∞ design (solid) and equiripple design (dotted)

Problem 2. *Given a stable, strictly proper $F(s)$, stable, proper $P(s)$, upsampling factor L, delay d, sampling period T, find $K(z)$ which minimizes*

$$\|\mathcal{E}\|_\infty := \sup_{w_c \in L^2} \frac{\|e_c\|_{L^2}}{\|w_c\|_{L^2}}.$$

Problem 2 is reducible to a finite-dimensional discrete-time optimization [5], and hence the optimal filter $K(z)$ can be obtained by a standard numerical computation software (e.g., MATLAB).

4. DESIGN EXAMPLE

We here present a design example of $\Delta\Sigma$ DA converters. The design parameters are as follows: the sampling period $T = 1$, the upsampling ratio $L = 8$, the reconstruction delay $d = T/L = 1/8$, and the analog filters are $P(s) = 1/(T_c s + 1)^2$, $T_c := T/\pi$ and

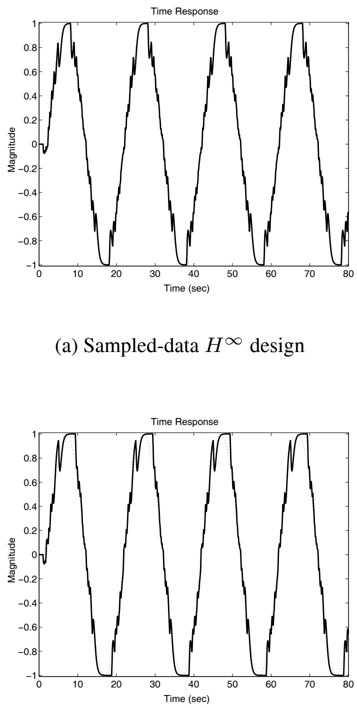
$$F(s) = \frac{1}{(Ts+1)(0.1Ts+1)}, \quad T = 22.05/\pi.$$

Note that the lowpass filter $F(s)$ simulates the frequency energy distribution of a typical orchestral music, which are observed by FFT analysis of analog records of some orchestral music. We design the FIR filter $R(z)$ (of order 7) in $\Delta\Sigma$ modulator by LMI [8], and the interpolation filter $K(z)$ by the sampled-data H^∞ optimization. For comparison, we take $R(z) = z^{-1}$ and the equiripple filter for $K(z)$ of order 21 as a conventional design.

The obtained interpolation filters are shown in Fig. 5. The gain around $\omega = 1/T_c = \pi$ of our filter is relatively large because the filter is designed by considering the lowpass characteristic of $P(s)$.

Then, we simulate the oversampling $\Delta\Sigma$ DA converter shown in Fig. 3 with a binary quantizer

$$Q(\psi) = \operatorname{sgn}(\psi) = \begin{cases} 1, & \psi \geq 0, \\ -1 & \psi < 0. \end{cases}$$



(a) Sampled-data H^∞ design

(b) Equiripple design

Fig. 6. Time response

Table 1. Comparison of error

	Sampled-data design	Conventional design
$\|e_c\|_\infty$	2.08×10^{-1}	2.67×10^{-1}
$\|e_c\|_2$	5.68×10^{-1}	7.21×10^{-1}
$\mathrm{RMS}(e_c)$	6.34×10^{-2}	8.06×10^{-2}

We take for the digital input u a sinusoidal wave $u[k] = \sin(0.1\pi k), \quad k = 0, 1, 2, \ldots, 80$. The time responses are shown in Fig. 6. The conventional DA converter shows large errors around $t = 10, 20, \ldots$. To see the difference, we show the absolute error in Fig. 7, and some norms of the error in Table 1. In the table, RMS is the root-mean-square values defined as follows: For fixed $T_f > 0$, $\mathrm{RMS}(e_c) := \left\{ \frac{1}{T_f} \int_0^{T_f} |e_c(t)|^2 dt \right\}^{\frac{1}{2}}$. These comparisons show that our design is superior to the conventional one.

5. CONCLUSIONS

We have proposed a new design method for $\Delta\Sigma$ modulators and oversampling $\Delta\Sigma$ DA converters via H^∞ optimization. We have presented design examples and shown the advantages of the present method.

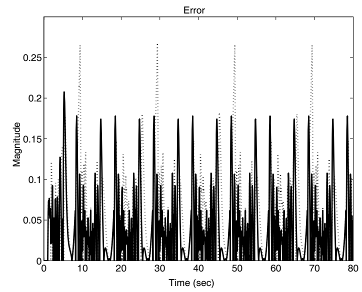


Fig. 7. Error: sampled-data H^∞ design (solid) and equiripple design (dotted)

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